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THE LAMINAR BOUNDARY LAYER AT HYPERSONIC SPEEDS

By Carlo Ferrari

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THE LAMINAR BOUNDARY LAYER AT HYPERSONIC SPEEDS*

By Carlo Ferrari**

SUMMARY

A theoretical inquiry is made into the nature of the laminar boundary layer on an airfoil immersed in a hypersonic stream under the assumptions that: (a) there is no heat transfer to the wall and the Prandtl number is unity and (b) there is a zero gradient of pressure normal to the direction of development of the layer along the wall.

The object of the first of these restrictions is to make it possible to take a mathematically more simple approach to the problem than would otherwise be possible if the complete general case were essayed, and yet the degree of approximation will be maintained on a par with that which is inherent in the statement of the basic differential equations themselves, which are governing the flow. That the second hypothesis is justifiable will be demonstrated in the course of working out the present analysis.

In the derivations given here, the treatment will be strictly applicable only at a sufficient distance downstream from the leading edge. To be more precise, the distance downstream at which the analysis begins to be valid must be great enough so that $M_\infty^2 / \sqrt{Re_x} < 1$, where M_∞ is the free-stream Mach number and Re_x denotes the local Reynolds number (which is based on the distance measured from the leading edge, and reaching downstream to the x-position in question at which the local Reynolds number is to be evaluated). It is further agreed that the present note is to confine attention solely to those cases in which the angular deviations in the flow are small. This is to say, it is assumed in this study that $M_\infty \beta < 1$ everywhere, where β is the local angular deviation of the velocity vector from the direction of the free-stream flow. The perturbation angle is the sum of both the deflection due to the shape of the solid profile over which the flow is coursing and the angular deflection brought about by the flow disturbances produced within the boundary layer.

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*"Sullo Strato Limite Laminare in Corrente Ipersonica." L'Aerotecnica, vol. XXXVI, no. 2, April 1956, pp. 68-94.

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The hypothesis is also made that no mixing occurs between the external isentropic flow and the internal viscous flow, so that for all intents and purposes the angle of inclination (measured with respect to the free-stream direction) of each streamline at the outer edge of the boundary layer does not differ from the angle generated there by the action of the viscous flow lying adjacent to the demarcation line between these two regions. Justification will be presented for use of this particularly simplifying assumption during the course of arriving at the salient propositions derived subsequently in the text. Under this hypothesis, then, the pressure distribution existing along the surface of the airfoil may be obtained when the shape of this contour is specified, or, better yet, when some governing parameters which characterize the contour shape are given. On the other hand, the shape of the profile may be determined when the pressure distribution that must exist along it is specified.

Once the pressures have been determined, then the skin friction is calculable. Numerical applications of the analytic methods adduced are made to illustrate the use of the theory in two different situations: (a) for flow along a flat plate and (b) for flow along a curved wall, the shape of which is specified by means of certain governing profile parameters.

On the basis of the results deduced from such an analysis, it becomes clear that the influence of the pressure gradient, created by the presence of the thick boundary layer, is appreciable even when the hypersonic similarity parameter χ_e is as low as 0.12. These severe alterations in the pressure perturbations influence in turn the character of the boundary layer to such a degree that the skin friction coefficient C_f can exhibit an increment of such great magnitude that a reversal in the trend of the curve of skin friction plotted against M_∞ can occur, as compared with what is usually found to occur if one neglects the effect of viscosity in distorting the pressure distributions.

INTRODUCTION

At hypersonic speeds the region of flow influenced by the pressure field created by a given obstacle in the stream is of the same order of magnitude as the region in which the viscous effects are important, so that, as a matter of fact, the entire perturbed area surrounding the obstacle should be treated as though it were a boundary layer. Under such conditions it is not permissible to continue to assume that the pressures found at the outer edge of the boundary layer will be the same as those existing along the surface of the obstacle, as is ordinarily done under the assumption of zero viscosity for most of the flow field, except for the thin layer close to the wall. On the contrary, under hypersonic conditions, when the subsonic and supersonic ranges of speed have been exceeded, it is necessary to account for the fact that the viscosity of the fluid permeates the flow and influences the pressure perturbations everywhere in the disturbed field. Thus the pressure distribution at the outer confines of the boundary layer must be derived in the course of the analysis. In order to carry out the determination of such a pressure distribution it is most convenient to tackle separately the two contrasting situations wherein: (a) $M_\infty\beta < 1$, where β is the local angular deflection of the velocity vector with respect to the free-stream flow, taking into account both the deflection due to the shape of the solid profile as well as the angular deviation produced by the flow within the boundary layer, and where M_∞ is the free-stream Mach number, or (b) $M_\infty\beta > 1$.

The first case represents the situation where the influence of the boundary layer on the pressure distribution in the external stream is going to be very weak, and it is only this type of flow which is to be examined in the present analysis. The other basic hypotheses to be premised here are that (1) the boundary layer is laminar, (2) there is no heat transfer to the constraining wall, and (3) the Prandtl number of the fluid is unity.

The case for which the deviations in the flow vector are allowed to be such that $M_\infty\beta > 1$ will be treated in a subsequent report. This situation may be characterized, in fact, as the one for which the influence of the boundary layer on the pressure distribution in the external stream is highly pronounced, or, in fact, it may be called the "strong interference" type of flow. In the proposed sequel report to this one, the effect of heat transfer will also be examined in some detail.

In regard to previous work in this field, one can cite the work of Lester Lees and Ronald F. Probstein (ref. 1), concerning the laminar boundary layer in hypersonic flow for the case of weak interference. Their analysis is somewhat prefactory because it is confined solely to examination of the flow over a flat plate, and they use a method of successive approximations, which presents obstacles to rapid calculation, especially if the desired degree of accuracy is narrowed to desirably strict limits. Likewise, precursive considerations of an approximate nature have also been made by others in an attempt to understand what takes place when there is a strong interaction produced in the external flow. Introductory remarks on this score, but confined solely to flow over a flat plate, have been made by Lester Lees (ref. 2), Shan-Fu Shen (ref. 3), and Ting-Yi Li and H. T. Nagamatsu (ref. 4).

In this present study the objectives are kept quite broad by allowing the contour of the constraining wall to have any shape whatsoever, so long as the restriction as to $M_\infty\beta$ being everywhere less than unity is not transgressed. With this understanding the intent here is to determine the pressure distribution along the wall when its shape is specified or when the geometric characteristics are designated by means of certain governing parameters; or on the other hand, by means of the method to be expounded here, one may also determine the shape of the wall when the pressure distribution is prescribed. Finally, it will be demonstrated how one can proceed to calculate the skin friction drag. Numerical applications are given to illustrate the theoretical methods expounded for two specific cases; the first for a wall-shape having specified parameters prescribing its geometry, while the second example pertains to a simple flat plate, used for comparative purposes.

1. LIST OF PRINCIPAL SYMBOLS

X coordinate axis taken in the direction of and having the sense of the undisturbed free-stream velocity, and so positioned that the origin is made to coincide with the leading edge of the airfoil

Y coordinate axis taken in the direction normal to the X-axis (it is taken for granted that the Y-coordinates of the constraining wall along which the boundary layer is coursing will always be close to the X-axis)

L reference length; $x = \frac{X}{L}$, and $y = \frac{Y}{L}$

U,V components of velocity taken in the direction of the X- and Y-axes, respectively

U_e velocity of the stream at the outer edge of the boundary layer (where it is assumed that the component of this velocity in the direction of the X-axis coincides, for all practical purposes, with the magnitude of the velocity itself existing at the confines of the boundary layer)

U_l limiting velocity attained when the flow expands into a vacuum

$u = \frac{U}{U_l}$, $u_\infty = \frac{U_\infty}{U_l}$, $u_e = \frac{U_e}{U_l}$

p pressure

ρ density

T absolute temperature

i enthalpy

μ coefficient of viscosity

ν kinematic viscosity

Note: When the symbols listed above carry the subscript e they refer to the values that these respective quantities take on at the outer edge of the boundary layer; and likewise, the subscript w pertains to values existing at the surface of the constraining wall, while the subscript ∞ denotes values pertaining to the free stream.

$\rho^* = \dfrac{\rho}{\rho_\infty}$

$\mu^* = \dfrac{\mu}{\mu_\infty}$

ψ stream function, defined by means of the differential relations

$$U = \frac{1}{\rho}\frac{\partial\psi}{\partial Y}; \quad V = -\frac{1}{\rho}\frac{\partial\psi}{\partial X}$$

$\psi^* = \dfrac{\psi}{\rho_\infty U_\infty L}; \ \overline{\psi}^* = \psi^* \dfrac{\sqrt{Re}}{2}$

Re Reynolds number referred to the undisturbed stream, $Re = \dfrac{U_\infty L}{\nu_\infty}$

Re_x local Reynolds number, $Re_x = Re \cdot x$

C constant which appears in the relation giving the dependence of the viscosity on temperature, when expressed affinely as $\dfrac{\mu}{\mu_e} = C\dfrac{T}{T_e}$

C_w constant appearing in the affine relation $\dfrac{\mu_w}{\mu_\infty} = C_w \dfrac{T_w}{T_\infty}$

C_e constant appearing as the proportionality factor in $\dfrac{\mu_\infty}{\mu_e} = C_e \dfrac{T_\infty}{T_e}$

γ adiabatic exponent

M_∞ free-stream Mach number

β_e inclination of the streamlines at the outer edge of the boundary layer, measured with respect to the **X**-axis

β_w slope of the constraining wall, measured with respect to the **X**-axis, taken at any arbitrary general point along its length

$$\beta_O = (\beta_w)_{x=O}$$

$$\varphi^* = \frac{1}{u_\infty} \int_0^x \frac{\rho_e}{\rho_\infty} u_e \, dx$$

$$\theta = \frac{\bar{\psi}^*}{\sqrt{\varphi^*}}$$

$$C^* = \frac{C}{C_e} = C_w$$

$$\chi_e = \frac{2C^* M_\infty^3}{\sqrt{Re}}$$

$$\chi_e' = M_\infty \beta_O$$

F_n^* function of θ, as defined by equation (20)

f_n^* function of θ, as defined by equation (20')

$$z = \frac{(1 - u^2)}{\left(1 - u_e^2\right)}$$

$$z_O = \frac{1}{\left(1 - u_e^2\right)}$$

$$Z = z - 1$$

$$Z_O = z_O - 1$$

$$A_O = \frac{\gamma - 1}{2} M_\infty^2$$

A_n^* constants defined through the relation that

$$Z_0 = A_0 + \sum_{n \gtrless 0} A_n^* (\varphi^*)^{\frac{n}{2}}$$

2. GOVERNING EQUATIONS

It is to be taken for granted that the gradient of pressures normal to the direction of development of the boundary layer along the wall is going to be zero in all cases now under consideration. Under this restriction, it then follows that the scalar equations expressing the momentum change and force balance are

$$\left.\begin{aligned} \rho U \frac{\partial U}{\partial X} + \rho V \frac{\partial U}{\partial Y} = -\frac{\partial p}{\partial X} + \frac{\partial}{\partial Y}\left(\mu \frac{\partial U}{\partial Y}\right) = \rho_e U_e \frac{d U_e}{d X} + \frac{\partial}{\partial Y}\left(\mu \frac{\partial U}{\partial Y}\right) \\ \frac{\partial p}{\partial Y} = 0 \end{aligned}\right\} \quad [1]$$

while, likewise, the following expression for the energy integral also holds.

$$i + \frac{1}{2} U^2 = i_e + \frac{1}{2} U_e^2 = \frac{1}{2} U_l^2. \quad [2]$$

From equation (2) it may be deduced that

$$\frac{T}{T_e} = \frac{1 - u^2}{1 - u_e^2} = \frac{\rho_e}{\rho} \quad [3]$$

as is demonstrated in appendix A. Furthermore, if one assumes that the viscosity is going to be affinely related to the temperature, according to the relation

$$\frac{\mu}{\mu_e} = C \frac{T}{T_e} \quad [4]$$

and if the nondimensional versions of the velocities and density, etc., as defined in the symbol list for u, u_e, ρ^*, etc., are inserted into the first of the relations presented in equations (1), then one may convert this momentum equation into the following form

$$\rho^* u \frac{\partial u}{\partial x} = \rho_e^* u_e \frac{du_e}{dx} + C \rho^* \rho_e^* \mu_e^* \frac{1}{2} \frac{1}{Re} \frac{u}{u_\infty} \frac{\partial^2 u}{\partial \psi^{*2}} \qquad [5]$$

by use of the Von Mises transformation (see p. 122 of ref. 5), where ψ represents the stream function, defined in the symbol list.

Now make the auxiliary transformations

$$\xi = \frac{1}{u_\infty} \int_0^x \frac{\rho_e^* \mu_e^*}{\sqrt{1 - u_e^2}} dx \quad ; \quad z = \frac{1 - u^2}{1 - u_e^2} \quad ; \quad \bar{\psi}^* = \frac{\sqrt{Re}}{2} \psi^* \qquad [6]$$

so the equation (5) becomes converted to

$$\frac{1}{C} \frac{1}{1 - u_e^2} \frac{\partial z}{\partial \xi} = \frac{\sqrt{z_0 - z}}{4} \frac{\partial^2 z}{\partial \bar{\psi}^{*2}} \qquad [7]$$

provided it is understood that

$$z_0 = \frac{1}{1 - u_e^2} \qquad [8]$$

Furthermore, let the additional transformation be introduced that

$$\begin{aligned} \xi^* &= C \int_0^\xi (1 - u_e^2)\, d\xi = \frac{C}{u_\infty} \int_0^x \rho_e^* \mu_e^* \sqrt{1 - u_e^2}\, dx \\ Z &= z - 1 = \frac{u_e^2 - u^2}{1 - u_e^2} \quad ; \quad Z_0 = z_0 - 1 = \frac{u_e^2}{1 - u_e^2} \end{aligned} \qquad [9]$$

from which it follows that equation (7) will now appear as

$$\frac{\partial Z}{\partial \xi^*} = \frac{1}{4} \sqrt{Z_0 - Z} \frac{\partial^2 Z}{\partial \bar{\psi}^{*2}} \qquad [10]$$

where the boundary conditions on the unknown function Z under these circumstances are

$$Z = 0 \text{ for } \bar{\psi}^* = \infty \quad ; \quad Z = Z_0 \text{ for } \bar{\psi}^* = 0 \qquad [11]$$

3. SOLUTION OF DIFFERENTIAL EQUATION GOVERNING THE FLOW IN THE OUTER PART OF THE BOUNDARY LAYER

In the vicinity of the outer edge of the boundary layer the differential equation governing the flow, equation (10), may be written in the more tractable form of

$$\frac{\partial Z}{\sqrt{Z_0}\,\partial \xi^*} = \frac{1}{4}\frac{\partial^2 Z}{\partial \bar{\psi}^{*2}} \qquad [12]$$

where this reduction is seen to be legitimate in consequence of the boundary condition, expressed as the first of equations (11), which must be applicable in the region under consideration.

Now furthermore, by making the definition that

$$\varphi^* = \int_0^{\xi^*} \sqrt{Z_0}\, d\,\xi^* = \frac{1}{u_\infty} C^* \int_0^x \frac{p_e}{p_\infty} u_e\, dx = C^* \bar{\varphi}^* \qquad [13]$$

provided it is assumed that

$$\frac{\mu_\infty}{\mu_e} = C_e \frac{T_\infty}{T_e} \quad ; \quad C^* = \frac{C}{C_e} \qquad [14]$$

the governing differential equation becomes even more simplified, to read now

$$\frac{\partial Z}{\partial \varphi^*} = \frac{1}{4}\frac{\partial^2 Z}{\partial \bar{\psi}^{*2}} \qquad [12']$$

and this expression is identical to the one which arises in the study of the boundary layer in incompressible flow.

If the limiting value of Z for $\bar{\psi}^* = 0$ is put into the form of

$$Z_0 = \frac{u_\infty^2}{1 - u_\infty^2} + A(\varphi^*) = \frac{\gamma - 1}{2} M_\infty^2 + A(\varphi^*) = A_0 + A(\varphi^*) \qquad [15]$$

where the shorthand notation has been introduced that

$$A_0 = \frac{\gamma - 1}{2} M_\infty^2 \qquad [16]$$

then it follows that the indicated form for the solution to the governing differential equation, equation (12'), is just

$$Z(\varphi^*, \bar{\psi}^*) = A_0\, erfc\left(\frac{\bar{\psi}^*}{\sqrt{\varphi^*}}\right) + \frac{\psi^*}{\sqrt{\pi}} \int_0^{\varphi^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^* - \varphi'}}}{(\varphi^* - \varphi')^{3/2}} A(\varphi')\, d\varphi' \qquad [17]$$

and this solution will satisfy the imposed boundary conditions stated in equations (11).

4. TRANSFORMATION OF THE INDICATED SOLUTION

Let the unspecified function $A(\varphi^*)$ be written in the form of a series development for which

$$A(\varphi^*) = \Sigma_{n \leq 0} A_n^* \varphi^{*\frac{n}{2}} = \Sigma_{n \leq 0} A_n(\varphi^*) \qquad [18]$$

with the understanding that equation (18) is valid solely for $\varphi^* > \varphi_0^*$ when $n < 0$, where φ_0^* is a suitable value of φ^*, and on the other hand,

$$A_n(\varphi^*) = \text{Constant} = A_n^* \quad \text{for} \quad n \leq 0 \quad \text{when} \quad \varphi^* < \varphi_0^*$$

Now, by use of this development given in equation (18), the formal solution presented as equation (17) becomes converted to the form

$$Z(\varphi^*, \bar{\psi}^*) = A_0\, erfc(\theta) + \Sigma_{n \leq 0} \varphi_0^{*\frac{n}{2}} F_n^*(\theta) \left(\frac{\varphi^*}{\varphi_0^*} \right)^{\frac{n}{2}} + \cdots \qquad [19]$$

provided one abides by the glosses now agreed upon and provided it is taken into account that a certain amount of approximation is introduced through calculation of the various integrals, which are treated in more detail in appendix B. The nonappearing terms which have been indicated by the three dots in equation (19) do not have to be specified for the purposes of this study, as is pointed out in the course of the observations made in appendix B. The rest of the nomenclature for the symbols appearing in equation (19) is assigned according to the following algebraic statements:

$$F_0^* = A_0^* \operatorname{erfc}(\theta) = A_0^* f_0^*$$

$$F_{-1}^* = A_{-1}^* e^{-\theta^2} = A_{-1}^* f_{-1}^*$$

$$F_{-2}^* = A_{-2}^* \left[\operatorname{erfc}(\theta) - \frac{1}{\pi} \log\left(\varphi_0^*/4\right) \theta e^{-\theta^2}\right] + A_{-3}^* \frac{2}{\sqrt{\pi}} \frac{1}{\varphi_0^{*1/2}} \theta e^{-\theta^2}$$

$$= A_{-2}^* f_{-2}^* + 2A_{-3}^* \frac{\theta e^{-\theta^2}}{\sqrt{\pi}\varphi_0^{*1/2}}$$

$$A_{-2}^* f_{-2}^* + 2A_{-3}^* f_{-2}'^*$$

$$F_{-3}^* = A_{-3}^* e^{-\theta^2}(1 - 2\theta^2) = A_{-3}^* f_{-3}^*$$

$$(F_n^*)_{n>0} = A_n^* \Gamma\left(1 + \frac{n}{2}\right) 4^{\frac{n}{2}}\, i^n\left[\operatorname{erfc}(\theta)\right] = A_n^* f_n^* \qquad (n > 0) \qquad [20]$$

where

$$\theta = \frac{\bar{\psi}^*}{\sqrt{\varphi^*}} \qquad [21]$$

and where the complement of the error function has been indicated by erfc θ, and where the repeated integration of this same function has been indicated by $i^n\left[\operatorname{erfc}(\theta)\right]$. It is considered to be sufficient for purposes of the present analysis to ignore the expressions for F_n^* having terms in A_n^* for which $n < -3$.

5. DEFLECTION OF THE STREAMLINES PRODUCED BY ACTION OF THE BOUNDARY-LAYER FLOW

By reference to the equation of continuity, it may be observed that the change in angle of inclination of the flow at the outer edge of the boundary layer, when measured with respect to the inclination that the stream has at the constraining wall itself, is given by means of the formula

$$\beta_e - \beta_w = \frac{2C^*}{\sqrt{Re}} \frac{u_e}{u_\infty} \frac{p_e}{p_\infty} \int_0^{\bar{\psi}^*_\delta} \frac{\partial}{\partial \varphi^*}\left(\frac{\rho_\infty u_\infty}{\rho u}\right) d\bar{\psi}^* = \frac{2C^*}{\sqrt{Re}} \frac{u_e}{u_\infty} \frac{p_e}{p_\infty} \frac{\partial}{\partial \varphi^*}\left[\sqrt{\varphi^*} \int_0^\infty \frac{\rho_e u_\infty}{\rho_e u_e}\left(\frac{\rho_e u_e}{\rho u} - 1\right) d\theta\right]$$
$$+ C^* \frac{u_e}{u_\infty} \frac{p_e}{p_\infty} \left[\frac{\partial}{\partial \varphi^*}\left(\frac{\rho_\infty u_\infty}{\rho_e u_e}\right)\right] \psi^*_\delta \qquad [22]$$

where $\bar{\psi}^*_\delta$ and ψ^*_δ denote, respectively, the values of $\bar{\psi}^*$ and of ψ^* which are obtained at the outer edge of the boundary layer. The upper limit on the integral appearing on the right-hand side of this equation is taken to be infinity, rather than the value of θ corresponding to $\bar{\psi}^* = \bar{\psi}^*_\delta$, because as θ increases into this range of values the integrand tends towards zero very rapidly.

6. CALCULATION OF THE INTEGRAND APPEARING IN THE ANGULAR-DEVIATION INTEGRAL

In order to carry out the determination of the angular deviation β_e it may be assumed that the expression for Z obtained as the external solution in section 4 will be valid here. The justification for this assumption will be given in appendix C. For purposes of evaluation in equation (22), it is thus possible to make the substitutions

$$\frac{\rho_e u_e}{\rho u} = (1+Z)\sqrt{\frac{Z_0}{Z_0 - Z}} \quad ; \quad \frac{u_e}{u_\infty} = \frac{1}{u_\infty}\sqrt{\frac{Z_0}{1+Z_0}} \qquad [23]$$

where Z is given by equation (19) and Z_0 is given by equation (18).

Furthermore, under the condition that $M_\infty \beta < 1$, it is admissible to assume that, at a sufficient distance downstream of the leading edge, the changes in state which take place along those streamlines which travel downstream to meet the outer edge of the boundary layer will take place isentropically in going from the conditions (p_∞, ρ_∞), existing in the undisturbed flow, to reach the conditions (p_e, ρ_e) at the outer edge of the boundary layer, regardless of the fact that a shock wave is traversed in the process. Taking into account this simplifying assumption, the density ratio is found from the expression

$$\frac{\rho_\infty}{\rho_e} = \left(\frac{p_\infty}{p_e}\right)^{\frac{1}{\gamma}} = \left(1 + A_0 \frac{u_\infty^2 - u_e^2}{u_\infty^2}\right)^{-\frac{1}{\gamma-1}} \qquad [24]$$

and inasmuch as

$$\frac{u_\infty^2 - u_e^2}{u_\infty^2} = 1 - \frac{1}{u_\infty^2}\frac{Z_0}{1+Z_0} = -\frac{A}{A_0}\frac{1}{1+A_0+A} \qquad [25]$$

it turns out that

$$\frac{\rho_\infty}{\rho_e} \cong \left(\frac{1}{1+\frac{A}{A_0}}\right)^{-\frac{1}{\gamma-1}} = 1 + \frac{1}{\gamma-1}\frac{A}{A_0} + \frac{2-\gamma}{2(\gamma-1)^2}\frac{A^2}{A_0^2} + \frac{2\gamma^2-7\gamma+6}{6(\gamma-1)^3}\frac{A^3}{A_0^3} + \cdots \qquad [26]$$

provided it is assumed that $A_0 \gg 1$. The convergence of this series development is assured simply by stipulating that $\left|\frac{A}{A_0}\right| < 1$ must hold.

It is also true that

$$\frac{u_e}{u_\infty} = \frac{1}{u_\infty}\sqrt{\frac{Z_0}{1+Z_0}} = \sqrt{\frac{1 + A/A_0}{1 + \frac{A}{1+A_0}}} \cong 1$$

where again it is premised that $A_0 \gg 1$.

Thus,

$$\frac{\rho_\infty u_\infty}{\rho_e u_e} = 1 + \frac{1}{\gamma-1}\frac{A}{A_0} + \frac{2-\gamma}{2(\gamma-1)^2}\frac{A^2}{A_0^2} + \frac{2\gamma^2-7\gamma+6}{6(\gamma-1)^3}\frac{A^3}{A_0^3} + \cdots \qquad [27]$$

Now take Z in the form

$$Z = A_0 F_0 + Z^*$$

where

$$F_0 = erfc\,(\theta) \qquad \text{and} \qquad Z^* = \Sigma_{n \geq 0} F_n^*(\theta)\,\varphi^{*\frac{n}{2}}$$

and observe that then

$$\frac{Z_0}{Z_0 - Z} = \frac{A_0 + A}{A_0 (1 - F_0)\left(1 + \dfrac{A - Z^*}{A_0 - A_0 F_0}\right)} \qquad [28]$$

Now whenever Φ is of the order of magnitude of $\frac{A}{A_0}$, it is possible to develop $(1 + \Phi)^{-1}$ in a power series, where $\Phi = \frac{A - Z^*}{A_0(1 - F_0)}$. Making use of such a development, one may recast equation (28) as

$$\frac{Z_0}{Z_0 - Z} = \frac{1}{1 - F_0}\left[1 + \frac{Z^* - A F_0}{A_0 (1 - F_0)} + \frac{(A - Z^*)(A F_0 - Z^*)}{A_0^2 (1 - F_0)^2} + \frac{(A - Z^*)^2 (Z^* - A F_0)}{A_0^3 (1 - F_0)^3} + \dots\right] \qquad [28']$$

and consequently,

$$\left(\frac{Z_0}{Z_0 - Z}\right)^{1/2} = \frac{1}{(1 - F_0)^{1/2}}\Bigg\{1 + \frac{1}{2}\,\frac{Z^* - A F_0}{A_0 (1 - F_0)} + \frac{(A F_0 - Z^*)\,[4 (A - Z^*) - (A F_0 - Z^*)]}{8 A_0^2 (1 - F_0)^2}$$
$$+ \frac{(Z^* - A F_0)\,[(Z^* - A F_0)^2 + 4\,(A - Z^*)\,(Z^* - A F_0) + 8\,(A - Z^*)^2]}{16 A_0^3 (1 - F_0)^3}\Bigg\} + \dots \qquad [29]$$

Combining these above-derived expressions, it results therefore that

$$\frac{\rho_\infty u_\infty}{\rho_e u_e}\left(\frac{\rho_e u_e}{\rho u} - 1\right) = \left[1 + \frac{1}{\gamma - 1}\,\frac{A}{A_0} + \frac{2 - \gamma}{2 (\gamma - 1)^2}\,\frac{A^2}{A_0^2} + \frac{2 \gamma^2 - 7 \gamma + 6}{6 (\gamma - 1)^3}\,\frac{A^3}{A_0^3} + \dots\right]$$
$$\cdot\left[-1 + \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}}\{\,\} + \frac{1}{(1 - F_0)^{1/2}}\left(Z^* + \frac{1}{2} Z^* \frac{Z^* - A F_0}{A_0 (1 - F_0)} + \frac{1}{8} Z^* \frac{(A F_0 - Z^*)[4 (A - Z^*) - (A F_0 - Z^*)]}{A_0^2 (1 - F_0^2)}\right)\right] \qquad [30]$$

where for brevity's sake the term standing on the right-hand side of equation (29) which is enclosed in the braces is indicated in equation (30) merely by use of the empty braces $\{\,\}$.

Now let the integrand of interest be represented by a sum of terms of the form

$$\frac{\rho_\infty u_\infty}{\rho_e u_e}\left(\frac{\rho_e u_e}{\rho u} - 1\right) = H_0 + H_1 + H_2 + H_3 + \dots \qquad [30']$$

wherein H_0 stands for a group of terms which are independent of A_n^*, H_1 represents a group of terms which contain A_n^*, H_2 represents a group containing A_n^{*2}, H_3 represents a group containing A_n^{*3}, and so forth. Thus, the definitions of the quantities H_i are the following:

$$H_0 = -1 + \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}};$$

$$H_1 = \frac{1}{2} \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}} \frac{Z^{\bullet} - A F_0}{A_0 (1 - F_0)} + \frac{Z^{\bullet}}{(1 - F_0)^{1/2}} + \frac{A}{A_0} \frac{1}{\gamma - 1} \left[-1 + \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}}\right];$$

$$\begin{aligned} H_2 = {} & \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}} \frac{(A F_0 - Z^{\bullet})[4(A - Z^{\bullet}) - (A F_0 - Z^{\bullet})]}{8 A_0^2 (1 - F_0)^2} + \frac{1}{2} \frac{Z^{\bullet}}{(1 - F_0)^{1/2}} \frac{Z^{\bullet} - A F_0}{A_0 (1 - F_0)} \\ & + \frac{1}{\gamma - 1} \frac{A}{A_0} \left[\frac{1 + A_0 F_0}{(1 - F_0)^{1/2}} \frac{1}{2} \frac{Z^{\bullet} - A F_0}{A_0 (1 - F_0)} + \frac{Z^{\bullet}}{(1 - F_0)^{1/2}}\right] + \frac{2 - \gamma}{2(\gamma - 1)^2} \frac{A^2}{A_0^2} \left[-1 + \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}}\right]; \end{aligned}$$

$$\begin{aligned} H_3 = {} & \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}} \frac{(Z^{\bullet} - A F_0)[(Z^{\bullet} - A F_0)^2 + 4(A - Z^{\bullet})(Z^{\bullet} - A F_0) + 8(A - Z^{\bullet})^2]}{16 A_0^3 (1 - F_0)^3} \\ & + \frac{1}{(1 - F_0)^{1/2}} \frac{1}{8} Z^{\bullet} \frac{(A F_0 - Z^{\bullet})[4(A - Z^{\bullet}) - (A F_0 - Z^{\bullet})]}{A_0^2 (1 - F_0)^2} \\ & + \frac{1}{\gamma - 1} \frac{A}{A_0} \left[\frac{1 + A_0 F_0}{(1 - F_0)^{1/2}} \frac{(A F_0 - Z^{\bullet})[4(A - Z^{\bullet}) - (A F_0 - Z^{\bullet})]}{8 A_0^2 (1 - F_0)^2} + \frac{1}{(1 - F_0)^{1/2}} \frac{Z^{\bullet}}{2} \frac{Z^{\bullet} - A F_0}{A_0 (1 - F_0)}\right] \\ & + \frac{2 - \gamma}{2(\gamma - 1)^2} \frac{A^2}{A_0^2} \left[\frac{1 + A_0 F_0}{(1 - F_0)^{1/2}} \frac{1}{2} \frac{Z^{\bullet} - A F_0}{A_0 (1 - F_0)} + \frac{Z^{\bullet}}{(1 - F_0)^{1/2}}\right] \\ & + \frac{2\gamma^2 - 7\gamma + 6}{6(\gamma - 1)^2} \frac{A^3}{A_0^3} \left[-1 + \frac{1 + A_0 F_0}{(1 - F_0)^{1/2}}\right]. \end{aligned} \qquad [31]$$

7. CALCULATION OF THE INTEGRAL APPEARING IN THE STREAMLINE DEVIATION FORMULA

Having obtained the sought series expression for the integrand appearing in the integral giving the deviation of the streamlines, it merely remains to perform the piecemeal integrations, according to the formula

$$\int_0^{\infty} \frac{\rho_{\infty} u_{\infty}}{\rho_e u_e} \left(\frac{\rho_e u_e}{\rho u} - 1\right) d\theta = \int_0^{\infty} (H_0 + H_1 + H_2 + H_3 + \ldots) d\theta = I_0 + I_1 + I_2 + I_3 + \ldots$$

Proceeding with these evaluations, one finds that

$$I_0 = \int_0^{\infty} H_0 \, d\theta = \int_0^{\infty} \left[-1 + \frac{1 + A_0 \, erfc(\theta)}{\sqrt{erfc(\theta)}}\right] d\theta \cong A_0 K_0 \qquad [32]$$

where

$$K_0 = \int_0^{\infty} \frac{erfc \, \theta}{\sqrt{erf \, \theta}} \, d\theta . \qquad [32']$$

Likewise, then

$$I_1 = \int_0^\infty H_1 d\theta = A_0 \sum_{n \lesseqgtr 0} \varphi^{*\frac{n}{2}} \left[\frac{1}{2A_0} \int_0^\infty \frac{1 + A_0 \operatorname{erfc} \theta}{\sqrt{\operatorname{erf} \theta}} \frac{F_n^*(\theta) - A_n^* \operatorname{erfc} \theta}{A_0 \operatorname{erf} \theta} d\theta + \int_0^\infty \frac{F_n^*(\theta)}{A_0 \sqrt{\operatorname{erf} \theta}} d\theta \right.$$

$$\left. + \frac{1}{\gamma - 1} \frac{A_n^*}{A_0} \int_0^\infty \frac{\operatorname{erfc} \theta}{\sqrt{\operatorname{erf} \theta}} d\theta \right] = A_0 \sum_{n \lesseqgtr 0} K_{1,n} \varphi^{*\frac{n}{2}} \qquad [33]$$

In the case where $n \neq -2$, one has that

$$K_{1,n} = \frac{A_n^*}{A_0} \left[\frac{1}{2A_0} \int_0^\infty \frac{1 + A_0 \operatorname{erfc} \theta}{\sqrt{\operatorname{erf} \theta}} \frac{f_n^*(\theta) - \operatorname{erfc} \theta}{\operatorname{erf} \theta} d\theta + \int_0^\infty \frac{f_n^*(\theta)}{\sqrt{\operatorname{erf} \theta}} d\theta \right.$$
$$\left. + \frac{1}{\gamma - 1} \int_0^\infty \frac{\operatorname{erfc} \theta}{\sqrt{\operatorname{erf} \theta}} d\theta \right] = \frac{A_n^*}{A_0} K_{1,n}^* \qquad [33\text{-}a]$$

whereas, when $n = -2$, then

$$K_{1,-2} = \frac{A^*_{-2}}{A_0} K^*_{1,-2} + \frac{A^*_{-3}}{A_0} \frac{2}{\sqrt{\pi}\,\varphi_0^{*\,1/2}} \int_0^\infty \left(\frac{1}{2A_0} \frac{1 + A_0 \operatorname{erfc} \theta}{\operatorname{erf} \theta} + 1 \right) \frac{\theta e^{-\theta^2}}{\sqrt{\operatorname{erf} \theta}} d\theta$$
$$= \frac{A^*_{-2}}{A_0} K^*_{1,-2} + \frac{A^*_{-3}}{A_0} K'_{1,-2} \qquad [33\text{-}b]$$

where

$$K'_{1,-2} = \frac{2}{\sqrt{\pi}} \frac{1}{\varphi_0^{*\,1/2}} \int_0^\infty \left(\frac{1}{2A_0} \frac{1 + A_0 \operatorname{erfc} \theta}{\operatorname{erf} \theta} + 1 \right) \frac{\theta e^{-\theta^2}}{\sqrt{\operatorname{erf} \theta}} d\theta . \qquad [33\text{-c}]$$

Continuing on, to evaluation of the next integral, it is found that

$$I_2 = \int_0^\infty H_2 d\theta = A_0 \Sigma_{n \lesseqgtr 0} \Sigma_{m \lesseqgtr 0} \varphi^{*\frac{n+m}{2}} \left\{ \frac{1}{8A_0} \int_0^\infty \frac{1 + A_0 \operatorname{erfc} \theta}{\sqrt{\operatorname{erf} \theta}} \frac{\operatorname{erfc} \theta\, A_n^* - F_n^*}{A_0 (\operatorname{erf} \theta)^2} \right.$$
$$\cdot \left[4 \frac{A_m^* - F_m^*}{A_0} - \frac{\operatorname{erfc} \theta \cdot A_m^* - F_m^*}{A_0} \right] d\theta + \frac{1}{2} \int_0^\infty \frac{F_n^*}{A_0 \sqrt{\operatorname{erf} \theta}} \frac{F_m^* - \operatorname{erfc} \theta\, A_m^*}{A_0 \operatorname{erf} \theta} d\theta$$
$$+ \frac{1}{\gamma - 1} \int_0^\infty \frac{A_n^*}{A_0} \left| \frac{1 + A_0 \operatorname{erfc} \theta}{2A_0 \sqrt{\operatorname{erf} \theta}} \frac{F_m^* - \operatorname{erf} \theta \cdot A_m^*}{A_0 \operatorname{erf} \theta} + \frac{F_m^*}{A_0 \sqrt{\operatorname{erf} \theta}} \right| d\theta \qquad [34]$$
$$\left. + \frac{2 - \gamma}{2(\gamma - 1)^2} \frac{A_n^* A_m^*}{A_0^2} \int_0^\infty \frac{\operatorname{erfc} \theta}{\sqrt{\operatorname{erf} \theta}} d\theta \right\} = A_0 \Sigma_{n \lesseqgtr 0} \Sigma_{m \lesseqgtr 0} \varphi^{*\frac{n+m}{2}} K_{2,n,m} .$$

In the case where $n \neq -2$ and $m \neq -2$, one has that

$$K_{2,n,m} = \frac{A_n^* A_m^*}{A_0^2} \left[\frac{1}{8 A_0} \int_0^\infty \frac{1 + A_0 \,\mathrm{erfc}\,\theta}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{\mathrm{erfc}\,\theta - f_n^*}{(\mathrm{erf}\,\theta)^2} \, [4(1 - f_m^*) - \mathrm{erfc}\,\theta - f_m^*)] \, d\theta \right.$$

$$+ \frac{1}{2} \int_0^\infty \frac{f_n^*}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{f_m^* - \mathrm{erfc}\,\theta}{\mathrm{erf}\,\theta} \, d\theta + \frac{1}{\gamma - 1} \int_0^\infty \left[\frac{1 + A_0 \,\mathrm{erfc}\,\theta}{2 A_0 \sqrt{\mathrm{erf}\,\theta}} \, \frac{f_m^* - \mathrm{erfc}\,\theta}{\mathrm{erf}\,\theta} + \frac{f_m^*}{\sqrt{\mathrm{erf}\,\theta}} \right] d\theta$$

$$\left. + \frac{2 - \gamma}{2(\gamma - 1)^2} \int_0^\infty \frac{\mathrm{erfc}\,\theta}{\sqrt{\mathrm{erf}\,\theta}} \, d\theta \right] = \frac{A_n^* A_m^*}{A_0^2} K^*_{2,n,m}. \qquad [34\text{-}a]$$

In the case where $n = -2$, $m \neq -2$, on the other hand, one finds that

$$K_{2,-2,m} = \frac{A^*_{-2} A_m^*}{A_0^2} K^*_{2,-2,m} - \frac{A_m^*}{A_0} \left(\frac{A^*_{-3}}{A_0} \frac{2}{\sqrt{\pi}\, \varphi_0^{*\,1/2}} \right) \cdot \left[\frac{1}{8 A_0} \int_0^\infty \frac{1 + A_0 \,\mathrm{erfc}\,\theta}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{\theta e^{-\theta^2}}{(\mathrm{erf}\,\theta)^2} [4(1 - f_m^*) - (\mathrm{erfc}\,\theta - f_m^*)] \, d\theta \right.$$

$$\left. - \frac{1}{2} \int_0^\infty \frac{\theta e^{-\theta^2}}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{f_m^* - \mathrm{erfc}\,\theta}{\mathrm{erf}\,\theta} \, d\theta \right] = \frac{A^*_{-2} A_m^*}{A_0^2} K^*_{2,-2,m} + \frac{A^*_{-3} A_m^*}{A_0^2} K'_{2,-2,m} \qquad [34\text{-}b]$$

whereas, for $n \neq -2$, $m = -2$, one sees that

$$K_{2,n,-2} = \frac{A_n^* A^*_{-2}}{A_0^2} K^*_{2,n,-2} - \frac{A_n^*}{A_0} \frac{A^*_{-3}}{A_0} \frac{2}{\sqrt{\pi}\, \varphi_0^{*\,1/2}} \left[\frac{3}{8 A_0} \int_0^\infty \frac{1 + A_0 \,\mathrm{erfc}\,\theta}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{\mathrm{erfc}\,\theta - f_n^*}{(\mathrm{erf}\,\theta)^2} \, \theta e^{-\theta^2} \, d\theta \right.$$

$$\left. - \frac{1}{2} \int_0^\infty \frac{f_n^*}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{\theta e^{-\theta^2}}{\mathrm{erf}\,\theta} \, d\theta - \frac{1}{\gamma - 1} \int_0^\infty \left(\frac{1 + A_0 \,\mathrm{erfc}\,\theta}{2 A_0 \sqrt{\mathrm{erf}\,\theta}} \, \frac{\theta e^{-\theta^2}}{\mathrm{erf}\,\theta} + \frac{\theta e^{-\theta^2}}{\sqrt{\mathrm{erf}\,\theta}} \right) d\theta \right] = \frac{A_n^* A^*_{-2}}{A_0^2} K^*_{2,n,-2}$$

$$+ \frac{A_n^* A^*_{-3}}{A_0^2} K'_{2,n,-2}. \qquad [34\text{-}c]$$

Finally, therefore, the fourth term in the evaluation of the expression for the streamline deviation is

$$I_3 = \int_0^\infty H_3 \, d\theta = A_0 \sum_{n \lesseqgtr 0} \sum_{m \lesseqgtr 0} \sum_{p \lesseqgtr 0} \varphi^{*\frac{n+m+p}{2}} \left\{ \frac{1}{16} A_0 \, \cdot \right.$$

$$\cdot \int_0^\infty \frac{1 + A_0 \,\mathrm{erfc}\,\theta}{\sqrt{\mathrm{erf}\,\theta}} \, \frac{F_n^* - \mathrm{erf}\,\theta \, A_n^*}{A_0 (\mathrm{erf}\,\theta)^3} \left[\left(\frac{F_m^*}{A_0} - \mathrm{erfc}\,\theta \, \frac{A_m^*}{A_0} \right) \left(\frac{F_p^*}{A_0} - \mathrm{erfc}\,\theta \, \frac{A_p^*}{A_0} \right) \right.$$

$$\left. + 4 \left(\frac{A_m^*}{A_0} - \frac{F_m^*}{A_0} \right) \left(\frac{F_p^*}{A_0} - \mathrm{erfc} \, \frac{A_p^*}{A_0} \right) + 8 \left(\frac{A_m^*}{A_0} - \frac{F_m^*}{A_0} \right) \left(\frac{A_p^*}{A_0} - \frac{F_p^*}{A_0} \right) \right] d\theta$$

$$+ \frac{1}{8} \int_0^\infty \frac{F_n^*}{A_0 \sqrt{\mathrm{erf}\,\theta}} \, \frac{\mathrm{erfc}\,\theta \cdot A_m^* - F_m^*}{A_0 (\mathrm{erf}\,\theta)^2} \left[4 \left(\frac{A_p^*}{A_0} - \frac{F_p^*}{A_0} \right) - \left(\mathrm{erfc}\,\theta \cdot \frac{A_p^*}{A_0} - \frac{F_p^*}{A_0} \right) \right] d\theta$$

$$+ \frac{1}{\gamma - 1} \frac{A_n^*}{A_0} \int_0^\infty \left\{ \frac{1 + A_0 \,\mathrm{erfc}\,\theta}{A_0 (\mathrm{erf}\,\theta)^{1/2}} \, \frac{\mathrm{erfc}\,\theta \, A_m^* - F_m^*}{8 A_0 (\mathrm{erf}\,\theta)^2} \left[4 \left(\frac{A_p^*}{A_0} - \frac{F_p^*}{A_0} \right) - \left(\frac{A_p^*}{A_0} \mathrm{erfc}\,\theta - \frac{F_p^*}{A_0} \right) \right] \right.$$

$$\left. + \frac{1}{2} \frac{F_m^*}{A_0 \sqrt{\mathrm{erf}\,\theta}} \, \frac{F_p^* - \mathrm{erfc}\,\theta \cdot A_p^*}{A_0 \,\mathrm{erf}\,\theta} \right\} d\theta + \frac{2 - \gamma}{2(\gamma - 1)^2} \frac{A_n^* A_m^*}{A_0^2} \cdot \int_0^\infty \left\{ \left[\frac{1 + A_0 \,\mathrm{erfc}\,\theta}{2 A_0 \sqrt{\mathrm{erf}\,\theta}} \, \frac{F_p^* - \mathrm{erfc}\,\theta \cdot A_p^*}{A_0 \,\mathrm{erf}\,\theta} \right. \right.$$

$$\left. \left. + \frac{F_p^*}{A_0 \sqrt{\mathrm{erf}\,\theta}} \right] d\theta + \frac{2\gamma^2 - 7\gamma + 6}{6(\gamma - 1)^2} \frac{A_n^* A_m^* A_p^*}{A_0^3} \cdot \int_0^\infty \frac{\mathrm{erfc}\,\theta}{\sqrt{\mathrm{erf}\,\theta}} \, d\theta \right\} = A_0 \sum_{n \lesseqgtr 0} \sum_{m \lesseqgtr 0} \sum_{p \lesseqgtr 0} \cdot \varphi^{*\frac{m+n+p}{2}} K_{3,n,m,p}. \qquad [35]$$

In the case where $n \neq -2$, $m \neq -2$, and $p \neq -2$, one finds that

$$K_{3,n,m,p} = \frac{A_n^* A_m^* A_p^*}{A_0^3} \Bigg\{ \frac{1}{16 A_0} \int_0^\infty \frac{1 + A_0\, erfc\, \theta}{\sqrt{erf\, \theta}} \frac{f_n^* - erfc\, \theta}{(erf\, \theta)^3} \cdot [(f_m^* - erfc\, \theta) \cdot (f_p^* - erfc\, \theta)$$

$$+ 4\,(1 - f_m^*)\,(f_p^* - erfc\, \theta) + 8\,(1 - f_m^*)\,(1 - f_p^*)]\, d\,\theta + \frac{1}{8} \int_0^\infty \frac{f_n^*}{\sqrt{erf\, \theta}} \frac{erfc\, \theta - f_m^*}{(erf\, \theta)^2}$$

$$\cdot\ [4\,(1 - f_p^*) - (erfc\, \theta - f_p^*)]\, d\,\theta + \frac{1}{\gamma - 1} \int_0^\infty \Bigg\{ \frac{1 + A_0\, erfc\, \theta}{A_0 \sqrt{erf\, \theta}} \frac{erfc\, \theta - f_m^*}{8\,(erf\, \theta)^2} [4\,(1 - f_p^*) - (erfc\, \theta - f_p^*)]$$

$$+ \frac{1}{2} \frac{f_m^*}{\sqrt{erf\, \theta}} \cdot \frac{f_p^* - erfc\, \theta}{erf\, \theta} \Bigg\} d\,\theta + \frac{2 - \gamma}{2\,(\gamma - 1)^2} \int_0^\infty \Bigg\{ \Bigg[\frac{1 + A_0\, erfc\, \theta}{2 A_0 \sqrt{erf\, \theta}} \frac{f_p^* - erfc\, \theta}{erf\, \theta} + \frac{f_p^*}{\sqrt{erf\, \theta}} \Bigg] d\,\theta$$

$$+ \frac{2\gamma^2 - 7\gamma + 6}{6\,(\gamma - 1)^2} \int_0^\infty \frac{erfc\, \theta}{\sqrt{erf\, \theta}}\, d\,\theta \Bigg\} = \frac{A_n^* A_m^* A_p^*}{A_0^3} K^*_{3,n,m,p}. \qquad [35\text{-}a]$$

In the case where $n = -2$, $m \neq -2$, and $p \neq -2$, it is found that

$$K_{3,-2,m,p} = \frac{A^*_{-2} A_m^* A_p^*}{A_0^3} K^*_{3,-2,m,p} + \frac{A_m^* A_p^*}{A_0^2} \left(\frac{A^*_{-3}}{A_0} \frac{2}{\sqrt{\pi}\, \varphi_0^{*1/2}} \right)$$

$$\cdot \Bigg\{ \frac{1}{16 A_0} \int_0^\infty \frac{1 + A_0\, erfc\, \theta}{\sqrt{erf\, \theta}} \frac{\theta e^{-\theta^2}}{(erf\, \theta)^3} [(f_m^* - erfc\, \theta)\,(f_p^* - erfc\, \theta) + 4\,(1 - f_m^*)\,(f_p^* - erfc\, \theta)$$

$$+ 8\,(1 - f_m^*)\,(1 + f_p^*)]\, d\,\theta + \frac{1}{8} \int_0^\infty \frac{\theta\, e^{-\theta^2}}{\sqrt{erf\, \theta}} \frac{erfc\, \theta - f_m^*}{(erf\, \theta)^2} [4\,(1 - f_p^*) - (erfc\, \theta - f_p^*)]\, d\,\theta$$

$$= \frac{A_m^* A_p^*}{A_0^2} \left(\frac{A^*_{-2}}{A_0} K^*_{3,-2,m,p} + \frac{A^*_{-3}}{A_0} K'_{3,-2,m,p} \right) \qquad [35\text{-}b]$$

where

$$K'_{3,-2,m,p} = \frac{2}{\sqrt{\pi}} \frac{1}{\varphi^{*1/2}} \Bigg\{ \frac{1}{16 A_0} \int_0^\infty \frac{1 + A_0\, erfc\, \theta}{(erf\, \theta)^{1/2}} \frac{\theta\, e^{-\theta^2}}{(erf\, \theta)^3} [(f_m^* - erfc\, \theta) \cdot (f_p^* - erfc\, \theta) + 4\,(1 - f_m^*)\,(f_p^* - erfc\, \theta)$$

$$+ 8\,(1 - f_m^*)\,(1 - f_p^*)]\, d\,\theta + \frac{1}{8} \int_0^\infty \frac{\theta\, e^{-\theta^2}}{\sqrt{erf\, \theta}} \frac{erfc\, \theta - f_m^*}{(erf\, \theta)^2} [4\,(1 - f_p^*) - (erfc\, \theta - f_p^*)]\, d\,\theta\,. \qquad [35\text{-}c]$$

In the case where $n \neq -2$, $m = -2$, and $p \neq -2$, it is likewise found that

$$K_{3,n,-2,p} = \frac{A_n^* A^*_{-2} A_p^*}{A_0^3} K^*_{3,n,-2,p} + \frac{A_n^* A_p^*}{A_0^2} \frac{A^*_{-3}}{A_0} \frac{2}{\sqrt{\pi}\, \varphi^{*1/2}}$$

$$\cdot \Bigg(\frac{1}{16 A_0} \int_0^\infty \frac{1 + A_0\, erfc\, \theta}{\sqrt{erf\, \theta}} \frac{f_n^* - erfc\, \theta}{(erf\, \theta)^3} [-3\,\theta\, e^{-\theta^2} (f_p^* - erfc\, \theta) - 8\,\theta\, e^{-\theta^2} (1 - f_p^*)]\, d\,\theta$$

$$+ \frac{1}{8} \int_0^\infty \frac{f_n^*}{\sqrt{erf\, \theta}} \frac{-\theta\, e^{-\theta^2}}{(erf\, \theta)^2} [4\,(1 - f_p^*) - erfc\, \theta - f_p^*)]\, d\,\theta + \frac{1}{\gamma - 1} \int_0^\infty \Bigg\{ \frac{1 + A_0\, erfc\, \theta}{A_0 \sqrt{erf\, \theta}} \frac{-\theta^2 e^{-\theta^2}}{8\,(erf\, \theta)^2}$$

$$[4\,(1 - f_p^*) - (erfc\, \theta - f_p^*)] + \frac{\theta\, e^{-\theta^2}}{2\sqrt{erf\, \theta}} \frac{f_p^* - erfc\, \theta}{erf\, \theta} \Bigg\} d\,\theta \Bigg) = \frac{A_n^* A_p^*}{A_0^2} \left(\frac{A^*_{-2}}{A_0} K^*_{3,n,-2,p} + \frac{A^*_{-3}}{A_0} K'_{3,n,-2,p} \right) \qquad [36]$$

where

$$K'_{3,n,-2,p} = \frac{2}{\sqrt{\pi}} \frac{1}{\varphi_0^{*1/2}} \left\{ \frac{1}{16 A_0} \int_0^\infty \frac{1 + A_0 \, erfc\,\theta}{\sqrt{erf\,\theta}} \frac{f_n^* - erfc\,\theta}{(erf\,\theta)^3} [-3\,\theta\, e^{-\theta^2} (f_p^* - erfc\,\theta) \right.$$

$$- 8\,\theta\, e^{-\theta^2} (1 - f_p^*)] \, d\theta + \frac{1}{8} \int_0^\infty \frac{f_n^*}{\sqrt{erf\,\theta}} \frac{-\theta\, e^{-\theta^2}}{(erf\,\theta)^2} [4(1 - f_p^*) - (erfc\,\theta - f_p^*)] \, d\theta$$

$$\left. + \frac{1}{\gamma - 1} \int_0^\infty \left\{ \frac{1 - A_0\, erfc\,\theta}{A_0 \sqrt{erf\,\theta}} \frac{-\theta\, e^{-\theta^2}}{8\,(erf\,\theta)^2} [4(1 - f_p^*) - (erfc\,\theta - f_p^*) - \frac{\theta\, e^{-\theta^2}}{2\sqrt{erf\,\theta}} \frac{f_p^* - erfc\,\theta}{erf\,\theta} \right\} d\theta. \right. \quad [36\text{-}a]$$

Finally, in the case where $n \neq -2$, $m \neq -2$, but $p = -2$, one has that

$$K_{3,n,m,-2} = \frac{A_n^* A_m^* A_{-2}^*}{A_0^3} K_{3,n,m,-2}^* + \frac{A_n^* A_m^*}{A_0^2} \frac{A_{-3}^*}{A_0} \frac{2}{\sqrt{\pi}} \frac{1}{\varphi_0^{*1/2}} \cdot \left\{ \frac{1}{16 A_0} \int_0^\infty \frac{1 + A_0 \text{ erfc } \theta}{\sqrt{\text{erf } \theta}} \frac{f_n^* - \text{erfc } \theta}{(\text{erf } \theta)^3} \Big[(-3 f_m^* \right.$$

$$- \text{erfc } \theta + 4)\theta e^{-\theta^2} - 8(1 - f_m^*)\theta e^{-\theta^2} \Big] d\theta + \frac{1}{8} \int_0^\infty \frac{f_n^*}{\sqrt{\text{erf } \theta}} \frac{\text{erfc } \theta - f_m^*}{(\text{erf } \theta)^2} (-3\theta e^{-\theta^2}) d\theta$$

$$+ \frac{1}{\gamma - 1} \int_0^\infty \left\{ \frac{1 + A_0 \text{ erfc } \theta}{A_0 \sqrt{\text{erf } \theta}} \frac{\text{erfc } \theta - f_m^*}{8(\text{erf } \theta)^2} (-3\theta e^{-\theta^2}) + \frac{1}{2} \frac{f_m^*}{\sqrt{\text{erf } \theta}} \frac{\theta e^{-\theta^2}}{\text{erf } \theta} \right\} d\theta$$

$$\left. + \frac{2\gamma}{2(\gamma - 1)^2} \int_0^\infty \left[\frac{1 + A_0 \text{ erfc } \theta}{2 A_0 \sqrt{\text{erf } \theta}} \frac{\theta e^{-\theta^2}}{\text{erf } \theta} + \frac{\theta e^{-\theta^2}}{\sqrt{\text{erf } \theta}} \right] d\theta \right\}$$

$$= \frac{A_n^* A_m^*}{A_0^2} \left(\frac{A_{-2}^*}{A_0} K_{3,n,m,-2}^* + \frac{A_{-3}^*}{A_0} K'_{3,n,m,-2} \right) \quad [37]$$

where

$$K'_{3,n,m,-2} = \frac{2}{\sqrt{\pi}} \frac{1}{\varphi_0^{*1/2}} \left\{ \ \right\} \quad [37\text{-}a]$$

in which the empty braces denote the series of expressions standing within the braces of equation (37).

The sought expression for the angular deviation integral is thus obtainable from the following series:

$$\sqrt{\varphi^*} \int_0^\infty \frac{\rho_\infty u_\infty}{\rho_e u_e} \left(\frac{\rho_e u_e}{\rho u} - 1 \right) d\theta = A_0 \Big[\sqrt{\varphi^*} K_0 + \Sigma_{n \gtrless 0} \varphi^{*\frac{n+1}{2}} K_{1,n}$$

$$+ \Sigma_{n \gtrless 0} \Sigma_{m \gtrless 0} \varphi^{*\frac{n+m+1}{2}} K_{2,n,m} + \Sigma_{n \gtrless 0} \Sigma_{m \gtrless 0} \Sigma_{p \gtrless 0} \varphi^{*\frac{n+m+p+1}{2}} K_{3,n,m,p} \Big]. \quad [38]$$

8. CALCULATION OF THE STREAM FUNCTION AT THE EDGE OF THE BOUNDARY LAYER

In order to evaluate the stream function at the edge of the boundary layer, it may be noted first of all that the velocity profile within the boundary layer may be expressed, through means of equation (9), in the following form:

$$\frac{u^2}{u_e^2} = 1 - \frac{Z}{Z_0} = \frac{erf(\theta) + \sum_{n \geq 0} \varphi^{*\frac{n}{2}} \left(-\frac{F_n^*}{A_0} + \frac{A_n^*}{A_0} \right)}{1 + \sum_{n \geq 0} \frac{A_n^*}{A_0} \varphi^{*\frac{n}{2}}} \qquad [39]$$

Now let it be assumed that the square of the velocity ratio $\left(\frac{u^2}{u_e^2}\right)$ will take on a constant value at the outer edge of the boundary layer, and let this constant be denoted by C_0, where C_0 will be close to unity. Thus it may be seen that the value of the stream function at the outer edge of the boundary layer has to obey the following relation, obtained from equations (39) by setting $\psi^* = \psi^*_\delta$:

$$(1 - C_0)\left(1 + \sum_n \frac{A_n^*}{A_0} \varphi^{*\frac{n}{2}}\right) = erfc(\theta_\delta) + \sum_n \frac{F_n^*(\theta_\delta)}{A_0} \varphi^{*\frac{n}{2}} \qquad [39']$$

where the quantity θ_δ is related to the stream function at the outer edge of the boundary layer ψ^*_δ, through the formula

$$\theta_\delta = \frac{1}{2} \psi^*_\delta \frac{\sqrt{Re}}{\sqrt{\varphi^*}}$$

where obviously θ_δ has been used to indicate the value of θ which, for each φ^*, corresponds to ψ^*_δ.

Having obtained the solution for θ_δ by solving equation (39') under variant parametric values for φ^*, it follows that the sought result for the stream function, when evaluated at the outer edge of the boundary layer, will be given by

$$\psi_{\delta^*} = \frac{2\sqrt{\varphi^*}}{\sqrt{Re}} \theta_\delta \, . \qquad [40]$$

When one compares the result just adduced, equation (39'), with the expression written above in equation (38) for the first term in the angular deviation formula, as given in full in equation (22), it is apparent that the second term on the right-hand side of this equation (22) is negligibly small with respect to the first one, when the Mach number is large, that is, when $A_0 \gg 1$. The neglect of the second term with respect to the first is countenanced on the secure grounds that, in fact, the ratio of the second term to the first is of the same order of magnitude as $\frac{1}{A_0}$.

9. CALCULATION OF THE COEFFICIENTS IN THE SERIES DEVELOPMENT FOR THE COMPLETE ENERGY INTEGRAL, CHARACTERIZED BY THE PARAMETER φ^*

On the basis of the notation and developments written out explicitly in the preceding sections, it may be seen that the derivative appearing in the first term (the only one needing to be evaluated) of the angular deviation now takes the form

$$\begin{aligned}
&\frac{\partial}{\partial \varphi^*}\left[\sqrt{\varphi^*}\int_0^\infty \frac{\rho_\infty u_\infty}{\rho_e u_e}\left(\frac{\rho_e u_e}{\rho u}-1\right)d\theta\right] = \frac{\partial}{\partial \varphi^*}\left\{A_0\left[\varphi^{*1/2} K_0 + \Sigma_{n \gtreqless 0}\, \varphi^{*\frac{1+n}{2}} K_{1,n}\right.\right.\\
&\left.\left.+ \Sigma_{n \gtreqless 0}\Sigma_{m \gtreqless 0}\, \varphi^{*\frac{1+n+m}{2}} K_{2,n,m} + \Sigma_{n \gtreqless 0}\Sigma_{m \gtreqless 0}\Sigma_{p \gtreqless 0}\, \varphi^{*\frac{1+n+m+p}{2}} K_{3,n,m,p}\right]\right\}\\
&= A_0\left[\frac{K_0}{2\varphi^{*1/2}} + \Sigma_{n \gtreqless 0}\frac{n+1}{2}\varphi^{*\frac{n-1}{2}} K_{1,n} + \Sigma_{n \gtreqless 0}\Sigma_{m \gtreqless 0}\frac{1+n+m}{2}\varphi^{*\frac{n+m-1}{2}} K_{2,n,m}\right.\\
&\left.+ \Sigma_{n \gtreqless 0}\Sigma_{m \gtreqless 0}\Sigma_{p \gtreqless 0}\frac{1+n+m+p}{2}\varphi^{*\frac{n+m+p-1}{2}} K_{3,n,m,p}\right].
\end{aligned}$$

In conformity and analogy with the assumption as to the isentropic nature of the processes taking place along the streamlines, as premised previously in setting down equation (24), it is likewise permissible to presume here, for purposes of obtaining a suitable expression for the pressure ratio entering equation (22), that now

$$\frac{p_e}{p_\infty} = \left[1 + A_0\frac{u^2_\infty - u_e^2}{u^2_\infty}\right]^{\frac{\gamma}{\gamma-1}} = 1 - \frac{\gamma}{\gamma-1}\frac{A}{A_0} + \frac{\gamma(2\gamma-1)}{2(\gamma-1)^2}\frac{A^2}{A_0^2} - \frac{\gamma(6\gamma^2-7\gamma+2)}{6(\gamma-1)^3}\frac{A^3}{A_0^3} + \cdots \qquad [41]$$

and inasmuch as $\frac{u_e}{u_\infty} \cong 1$, it follows that the sought angular deflection values are expressible as

$$
\begin{aligned}
\beta_e - \beta_w = {} & \frac{2\,C^{*} A_0}{\sqrt{Re}} \left[1 - \frac{\gamma}{\gamma - 1} \frac{A}{A_0} + \frac{\gamma\,(2\gamma - 1)}{2\,(\gamma - 1)^2} \frac{A^2}{A_0^2} - \frac{\gamma\,(6\gamma^2 - 7\gamma + 2)}{6\,(\gamma - 1)^3} \frac{A^3}{A_0^3} + \cdots \right] \\
& \cdot \left[\frac{K_0}{2\,\varphi^{*1/2}} + \Sigma_{n \lessgtr 0} \frac{1+n}{2} \varphi^{*\frac{n-1}{2}} K_{1,n} + \Sigma_{n \lessgtr 0} \Sigma_{m \lessgtr 0} \frac{n+m+1}{2} \varphi^{*\frac{n+m-1}{2}} K_{2,n,m} \right. \\
& \left. + \Sigma_{n \lessgtr 0} \Sigma_{m \lessgtr 0} \Sigma_{p \lessgtr 0} \frac{n+m+p+1}{2} \varphi^{*\frac{n+m+p-1}{2}} K_{3,n,m,p} \right] \\
= {} & \frac{2\,A_0 C^{*}}{\sqrt{Re}} \left\{ \frac{K_0}{2\,\varphi^{*1/2}} + \Sigma_{n \lessgtr 0} \left(\frac{1+n}{2} K_{1,n} - \frac{\gamma}{\gamma - 1} \frac{A_n^{*}}{A_0} \frac{K_0}{2} \right) \varphi^{*\frac{n-1}{2}} \right. \\
& + \Sigma_{n \lessgtr 0} \Sigma_{m \lessgtr 0} \left[\frac{\gamma\,(2\gamma - 1)}{2\,(\gamma - 1)^2} \frac{K_0}{2} \frac{A_n^{*} A_m^{*}}{A_0^2} - \frac{\gamma}{\gamma - 1} \frac{1+m}{2} K_{1,m} \frac{A_n^{*}}{A_0} \right. \\
& \left. + \frac{1+n+m}{2} K_{2,n,m} \right] \varphi^{*\frac{n+m-1}{2}} + \Sigma_{n \lessgtr 0} \Sigma_{m \lessgtr 0} \Sigma_{p \lessgtr 0} \left[- \frac{\gamma\,(6\gamma^2 - 7\gamma + 2)}{6\,(\gamma - 1)^3} \frac{K_0}{2} \frac{A_n^{*} A_m^{*} A_p^{*}}{A_0^3} \right. \\
& \left. \left. + \frac{\gamma\,(2\gamma - 1)}{2\,(\gamma - 1)^2} \frac{1+p}{2} K_{1,p} \frac{A_n^{*} A_m^{*}}{A_0^2} - \frac{\gamma}{\gamma - 1} \frac{p+m+1}{2} K_{2,m,p} \frac{A_n^{*}}{A_0} + \frac{n+m+p+1}{2} K_{3,n,m,p} \right] \varphi^{*\frac{n+m+p-1}{2}} \right\} + \cdots
\end{aligned}
\qquad [42]
$$

Now let it be considered that the angular inclination for the constraining wall will be developable in a series of the form

$$
\beta_w = \beta_0 + \beta_1\,\varphi^{*1/2} + \beta_2\,\varphi^{*} + \cdots\cdots = \beta_0 \Sigma_{i \geq 0}\, b_i\, \varphi^{*\frac{i}{2}} \qquad (b_0 = 1) \qquad [43]
$$

so that one may state, in consequence, that the angular inclination obtained at the outer edge of the boundary layer will be given by the expression

$$
\beta_e = \frac{2\,A_0 C^{*}}{\sqrt{Re}} \left\{ \right\} + \beta_0 \Sigma_i\, b_i\, \varphi^{*\frac{i}{2}} \qquad [44]
$$

where the empty braces standing in the first term on the right-hand side of this formula have been employed to denote the entire set of terms contained within the braces appearing in equation (42).

Under the dictates of the premises so far made in this analysis, it is true that the pressure ratio may be expressed as

$$
\frac{p_e}{p_\infty} = \left(1 + \frac{\gamma - 1}{2} M_\infty\, \beta_e \right)^{\frac{2\gamma}{\gamma - 1}} \qquad [45]
$$

and, thence, by referring back to equation (41), it is seen that

$$
1 + \frac{\gamma - 1}{2} M_\infty\, \beta_e = \left(1 + \frac{A}{A_0} \right)^{-\frac{1}{2}} = 1 - \frac{1}{2} \frac{A}{A_0} + \frac{3}{8} \frac{A^2}{A_0^2} - \frac{5}{16} \frac{A^3}{A_0^3} + \cdots
$$

from whence it follows that the angular inclination at the outer edge of the boundary layer may be written specifically as

$$\beta_e = \frac{1}{\gamma - 1}\,\frac{1}{M_\infty}\,\frac{A}{A_0}\left(-1 + \frac{3}{4}\,\frac{A}{A_0} - \frac{5}{8}\,\frac{A^2}{A_0^{\,2}} + \dots\right) = \frac{1}{\gamma-1}\,\frac{1}{M_\infty}\left[-\sum_{n \lesseqgtr 0} \frac{A_n^*}{A_0}\,\varphi^{*\frac{n}{2}}\right.$$
$$\left. + \frac{3}{4}\sum_{n \lesseqgtr 0}\sum_{m \lesseqgtr 0} \frac{A_n^*}{A_0}\,\frac{A_m^*}{A_0}\,\varphi^{*\frac{n+m}{2}} - \frac{5}{8}\sum_{n \lesseqgtr 0}\sum_{m \lesseqgtr 0}\sum_{p \lesseqgtr 0} \frac{A_n^*\,A_m^*\,A_p^*}{A_0^{\,3}}\,\varphi^{*\frac{n+m+p}{2}}\right] = \frac{2\,A_0\,C^*}{\sqrt{Re}}\Big\{\Big\} + \beta_0 \sum b_i\,\varphi^{*\frac{i}{2}} \qquad [46]$$

The solution for the coefficients in the series development for the complete energy integral characterized by the parameter φ^* may thus be effected by equating like terms in the power series developments for φ^* involved on the right- and left-hand sides of equation (46). Upon carrying out this matching process it is found that the following set of relationships result:

In the case of equating the coefficients of $(\varphi^*)^{-1/2}$, one obtains

$$\frac{1}{\gamma-1}\,\frac{1}{M_\infty}\left[-\frac{A^*_{-1}}{A_0} + \frac{3}{4}\sum_{n \lesseqgtr 0}\frac{A_n^*}{A_0}\,\frac{A^*_{-1-n}}{A_0} - \frac{5}{8}\sum_{n \lesseqgtr 0}\sum_{m \lesseqgtr 0}\frac{A_n^*\,A_m^*\,A^*_{-1-m-n}}{A_0^{\,3}}\right]$$
$$= \frac{2\,A_0\,C^*}{\sqrt{Re}}\left\{\frac{K_0}{2} + \frac{1}{2}\,K_{1,0} - \frac{\gamma}{\gamma-1}\,\frac{A_0^*}{A_0}\,\frac{K_0}{2} + \sum_{n \lesseqgtr 0}\left[\frac{\gamma\,(2\gamma-1)}{2\,(\gamma-1)^2}\,\frac{K_0}{2}\,\frac{A_n^*\,A^*_{-n}}{A_0^{\,2}}\right.\right.$$
$$\left. - \frac{\gamma}{\gamma-1}\,\frac{1-n}{2}\,K_{1,-n}\,\frac{A_n^*}{A_0} + \frac{1}{2}\,K_{2,n,-n}\right] + \sum_{n \lesseqgtr 0}\sum_{m \lesseqgtr 0}\left[-\frac{\gamma\,(6\gamma^2 - 7\gamma + 2)}{6\,(\gamma-1)^3}\,\frac{K_0}{2}\right.$$
$$\cdot\,\frac{A_n^*\,A_m^*\,A^*_{-n-m}}{A_0^{\,3}} + \frac{\gamma\,(2\gamma-1)}{2\,(\gamma-1)^2}\,\frac{1-n-m}{2}\,K_{1,-n,-m}\,\frac{A_n^*\,A_m^*}{A_0^{\,2}}$$
$$\left.\left. - \frac{\gamma}{\gamma-1}\,\frac{1-n}{2}\,K_{2,m,-n,-m}\,\frac{A_n^*}{A_0} + \frac{1}{2}\,K_{3,n,m,-n,-m}\right]\right\} \qquad [47]$$

whereas, when the general terms in $\varphi^{*q/2}$ are equated, one obtains the set of expressions

$$\frac{1}{\gamma-1}\,\frac{1}{M_\infty}\left[-\frac{A_q^*}{A_0} + \frac{3}{4}\sum_{n \lesseqgtr 0}\frac{A_n^*}{A_0}\,\frac{A^*_{q-n}}{A_0} - \frac{5}{8}\sum_{n \lesseqgtr 0}\sum_{m \lesseqgtr 0}\frac{A_n^*\,A_m^*\,A^*_{q-n-m}}{A_0^{\,3}}\right]$$
$$= \frac{2\,A_0\,C^*}{\sqrt{Re}}\left\{\frac{2+q}{2}\,K_{1,1+q} - \frac{\gamma}{\gamma-1}\,\frac{K_0}{2}\,\frac{A^*_{1+q}}{A_0} + \sum_{n \lesseqgtr 0}\left[\frac{\gamma\,(2\gamma-1)}{2\,(\gamma-1)^2}\,\frac{K_0}{2}\,\frac{A_n^*}{A_0}\,\frac{A^*_{1+q-n}}{A_0}\right.\right.$$
$$\left. - \frac{\gamma}{\gamma-1}\,\frac{2+q-n}{2}\,K_{1,1+q-n}\,\frac{A_n^*}{A_0} + \frac{2+q}{2}\,K_{2,n,1+q-n}\right] + \sum_{n \lesseqgtr 0}\sum_{m \lesseqgtr 0}\left[-\frac{\gamma\,(6\gamma^2-7\gamma+2)}{6\,(\gamma-1)^3}\,\frac{K_0}{2}\right.$$
$$\cdot\,\frac{A_n^*\,A_m^*\,A^*_{1+q-n-m}}{A_0^{\,3}} + \frac{\gamma\,(2\gamma-1)}{2\,(\gamma-1)^2}\,\frac{2+q-n-m}{2}\cdot K_{1,1+q-n-m}\,\frac{A_n^*\,A_m^*}{A_0^{\,2}} - \frac{\gamma}{\gamma-1}\,\frac{2+q-n}{2}$$
$$\left.\left.\cdot\,K_{2,m,1+q-n-m}\,\frac{A_n^*}{A_0} + \frac{2+q}{2}\,K_{3,n,m,1+q-n-m}\right]\right\} + b_q\,\beta_0 \qquad [48]$$

where $b_q = 0$ for $q < 0$.

It is apposite to introduce at this point the hypersonic similarity parameters

$$\chi_e = \frac{2\,M^3_\infty\,C^*}{\sqrt{Re}} \quad ; \quad \chi'_e = M_\infty\,\beta_0 \qquad [49]$$

and thus the various unknown coefficients which are to be evaluated by use of the equalities given in equations (47) and (48) may be expressed in terms of power series developments in the parameters χ_e and χ_e'. The working form for such solutions is given in the following array:

$$
\begin{aligned}
\frac{A_0^*}{A_0} &= D_0 \quad \chi_e^2 - (\gamma - 1)\chi'_e + \frac{3}{4}(\gamma-1)^2\chi'^2_e - \frac{1}{2}(\gamma-1)^3\chi'^3_e + \dots \\
\frac{A_2^*}{A_0} &= D_2 \quad \chi_e^2 - (\gamma - 1) b_2\chi'_e + \frac{3}{2}(\gamma-1)^2 b_2\chi'^2_e - \frac{3}{2}(\gamma-1)^3 b_2\chi'^3_e + \dots \\
\frac{A_4^*}{A_0} &= D_4 \quad \chi_e^2 - (\gamma - 1) b_4\chi'_e + \frac{3}{4}(b_2^2 + 2b_4)(\gamma-1)^2\chi'^2_e - \frac{3}{2}(b_2^2 + b_4)(\gamma-1)^3\chi'^3_e + \dots \\
\frac{A^*_{-1}}{A_0} &= D'_{-1}\,\chi_e + D''_{-1}\,\chi_e^3 + \dots \\
\frac{A^*_{-2}}{A_0} &= D_{-2}\,\chi_e^2 + \dots \\
\frac{A^*_{-3}}{A_0} &= D_{-3}\,\chi_e^3 + \dots \\
\frac{A_3^*}{A_0} &= D'_3\,\chi_e + D''_3\,\chi_e^3 + \dots
\end{aligned}
\tag{50}
$$

wherein the unwritten terms contain powers of χ_e and χ_e' with exponent larger than 3, while, in addition, when i is odd the terms b_i are taken to be zero. It may be noted that the coefficients D will turn out to be functions of γ and of χ_e', and the way these parameters may be evaluated will now be demonstrated.

Let the auxiliary definitions be introduced that

$$
\begin{aligned}
L_1 &= -(\gamma-1)\chi_e' + \frac{3}{4}(\gamma-1)^2\chi_e'^2 - \frac{1}{2}(\gamma-1)^3\chi_e'^3 \\
L_2 &= -(\gamma-1) b_2\chi_e' + \frac{3}{2}(\gamma-1)^2 b_2\chi_e'^2 - \frac{3}{2}(\gamma-1)^2 b_2\chi_e'^3 \\
L_3 &= -(\gamma-1) b_4\chi_e' + \frac{3}{4}(b_2^2 + 2b_4)(\gamma-1)^2\chi_e'^2 - \frac{3}{2}(b_2^2 + b_4)(\gamma-1)^3\chi_e'^3
\end{aligned}
\tag{51}
$$

and also let the symbols $\alpha_{i,h}$ be specified according to the following array, constituting equation (52):

α_{ik}	α_{i1}	α_{i2}	α_{i3}	α_{i4}	α_{i5}
α_{1k}	$4.24 - 0.67 \frac{\gamma}{\gamma-1}$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 5.945 \frac{\gamma}{\gamma-1} + 5.86$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 6.585 \frac{\gamma}{\gamma-1} + 5.295$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 4.445 \frac{\gamma}{\gamma-1}$	—
α_{2k}	$7.44 - 0.67 \frac{\gamma}{\gamma-1}$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 9.785 \frac{\gamma}{\gamma-1} + 13.965$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 10.165 \frac{\gamma}{\gamma-1} + 13.53$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 8.875 \frac{\gamma}{\gamma-1} + 15.105$	—
α_{3k}	$10.05 - 0.67 \frac{\gamma}{\gamma-1}$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 12.395 \frac{\gamma}{\gamma-1} + 20.95$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 13.80 \frac{\gamma}{\gamma-1} + 20.25$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 13.115 \frac{\gamma}{\gamma-1} + 21.475$	—
α_{4k}	$0.335 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} + 2.345 \frac{\gamma}{\gamma-1} + 2.93$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} + 4.24 \frac{\gamma}{\gamma-1} + 6.28$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} + 5.655 \frac{\gamma}{\gamma-1} + 0.83$	—	—
α_{5k}	$5.925 - 0.67 \frac{\gamma}{\gamma-1}$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 8.27 \frac{\gamma}{\gamma-1} + 9.86$	$0.335 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 4.24 \frac{\gamma}{\gamma-1} + 4.87$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 7.44 \frac{\gamma}{\gamma-1} + 10.71$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 8.605 \frac{\gamma}{\gamma-1} + 2.86$
α_{6k}	$0.67 \frac{\gamma}{\gamma-1}$	$-0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} + 2.345 \frac{\gamma}{\gamma-1} + 7.075$	—	—	—
α_{7k}	$-10.27 + 0.67 \frac{\gamma}{\gamma-1}$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 2.075 \frac{\gamma}{\gamma-1} - 1.35$	$0.335 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} + 8.74$	—	—
α_{8k}	$8.875 - 0.67 \frac{\gamma}{\gamma-1}$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 11.22 \frac{\gamma}{\gamma-1} + 17.76$	$0.67 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 11.68 \frac{\gamma}{\gamma-1} + 17.90$	$0.335 \frac{\gamma(2\gamma-1)}{(\gamma-1)^2} - 5.925 \frac{\gamma}{\gamma-1} + 8.3$	—

Also let the simplification of notation be made that

$$\Delta = 1 - \frac{3}{2} L_1 + \frac{15}{8} L^2 \quad ; \quad \alpha_1 = \frac{3}{2}\left(1 - \frac{15}{2} L_1\right) \quad ; \quad \alpha_2 = \frac{3}{2}\left(L_3 - \frac{5}{2} L_1 L_3 - \frac{5}{4} L_2^2\right). \qquad [53]$$

Consequently, the sequence of D-values will be determinable from the relationships given in equation (54).

$$D'_{-1} = -\frac{(\gamma-1)^2}{2}\,\frac{0.67 + L_1^2\alpha_{4.1}}{\Delta}$$

$$D'_1 = \frac{\alpha_1 L_2 D'_{-1} - \frac{(\gamma-1)^2}{2}(\alpha_{5.1} + L_1\alpha_{5.2})L_2}{\Delta}$$

$$D_{-2} = \frac{\frac{1}{2}\alpha_1 (D'_{-1})^2 + \frac{(\gamma-1)^2}{2}(\alpha_{6.1} + L_1\alpha_{6.2})D'_{-1}}{\Delta}$$

$$D_0 = \frac{\alpha_1(L_1 D_{-2} + D'_{-1}D'_1) - \frac{15}{8}L_2(D'_{-1})^2 - \frac{(\gamma-1)^2}{2}\left[L_2 D'_{-1}\alpha_{1.2} + D'_1(\alpha_{1.1} + L_1\alpha_{1.3})\right]}{\Delta}$$

$$D'_3 = \frac{\alpha_1 L_2 D'_1 + \alpha_2 D'_{-1} - \frac{(\gamma-1)^2}{2}(L_3\alpha_{8.1} + L_1 L_3\alpha_{8.2} + L_2^2\alpha_{8.4})}{\Delta}$$

$$D_2 = \frac{\alpha_1\left(L_2 D_0 + \frac{1}{2}D'^2_1 + D'_{-1}D'_3\right) + \frac{\alpha}{2}D_0 - \frac{15}{8}(L_1 D'^2_1 + 2L_2 D'_{-1}D'_1)}{\Delta} - \frac{(\gamma-1)^2}{2}\,\frac{D'_3(\alpha_{2.1} + \alpha_{2.2}) + L_2 D'_1\alpha_{2.3} + L_3 D'_{-1}\alpha_{2.4}}{\Delta}$$

$$D_4 = \frac{\alpha_1(L_2 D_2 + D'_1 D'_3) + \alpha_2 D_0 - \frac{15}{8}(L_2 D'^2_1 + 2L_2 L_3 D_{-2} + 2L_2 D'_{-1}D'_3 + 2L_3 D'_{-1}D'_1)}{\Delta} - \frac{(\gamma-1)^2}{2}\,\frac{L_2 D'_3\alpha_{3.3} + L_3 D'_1\alpha_{3.4}}{\Delta} \qquad [54]$$

$$D_{-3} = \frac{\alpha_1(D'_{-1}D_2) - \frac{5}{8}D'^3_{-1}\ \frac{(\gamma-1)^2}{2}(D_{-2}\alpha_{7.2} + L_1 D_{-2}\alpha_{7.2} + D'^2_{-1}\alpha_{7.3})}{\Delta}$$

$$D''_{-1} = \frac{\alpha_1(L_2 D_{-3} + D'_1 D_{-2}) + \frac{3}{2}D_0 D'_{-1} - \frac{15}{8}(D'^2_{-1}D'_1 - 2L_2 D'_{-1}D_{-2})}{\Delta} - \frac{(\gamma-1)^2}{2}\,\frac{2L_1 D_0\alpha_{4.1} + D'_{-1}D'_1\alpha_{4.2} + L_2 D_{-3}\alpha_{4.3}}{\Delta}$$

$$D''_1 = \frac{\alpha_1(L_2(D''_{-1} + D_0 D'_1 + D_2 D'_{-1} + D_{-2}D'_3) + \alpha_2 D_{-3}}{\Delta} - \frac{\frac{15}{8}\left[2(L_2 + L_3)D'_{-1}D_{-2} + 2L_2(D'_1 D_{-2} + D_0 D'_{-1}) + (D'_{-1})^2 D'_3 + (D'_1)^2 D'_{-1}\right]}{\Delta} - \frac{(\gamma-1)^2}{2}\,\frac{D_2\alpha_{5.1} + (L_1 D_2 + D_0 L_2)\alpha_{5.2} + (D'_1)^2\alpha_{5.3} + D'_{-1}D'_3\alpha_{5.4} + L_3 D_{-2}\alpha_{5.5}}{\Delta}$$

$$D''_3 = \frac{\alpha_1(L_3 D''_1 + D_2 D'_1 + D_4 D'_{-1} + D_0 D'_3) + \alpha_2 D''_{-2}}{\Delta} - \frac{15}{4}\,\frac{L_2(D_2 D'_{-1} + D_0 D'_1 + D_{-2}D'_3) + L_3(D_0 D'_{-1} + D'_1 D_{-2}) + D'_{-1}D'_1 D'_3 + L_2 L_3 D_{-3}}{\Delta} - \frac{(\gamma-1)^2}{2}\,\frac{D_4\alpha_{8.1} + (L_1 D_4 + L_3 D_0)\alpha_{8.2} + D'_1 D'_3\alpha_{8.3} + 2L_2 D_2\alpha_{8.4}}{\Delta}.$$

10. DEPENDENCE OF THE GEOMETRIC CONTOUR OF THE AIRFOIL ON THE PRESSURE DISTRIBUTION MAINTAINED ALONG THE EDGE OF THE BOUNDARY LAYER

It is seen that equations (50) give a relationship between the coefficients $\frac{A_n^*}{A_0}$ (which control what the pressures are at the edge of the boundary layer) and the parameters b (which determine the shape of the airfoil constituting the constraining wall). Thus, if the coefficients $\frac{A_n^*}{A_0}$ are known, because it is taken for granted that these values were fixed beforehand, then it results that the coordinates of the surface constituting the constraining wall will be produced by the analysis, inasmuch as one has, for known $\frac{A_n^*}{A_0}$ coefficients, that

$$C^*(x - x_0) = \int_{\varphi_0^*}^{\varphi} \left[1 + \Sigma_{n \lessgtr 0} \frac{A_n^*}{A_0} \varphi^{*\frac{n}{2}}\right]^{\frac{\gamma}{\gamma-1}} d\varphi^* \qquad [55]$$

$$C^* x_0 = \left[1 + \Sigma_n \frac{A_n^*}{A_0} \varphi_0^{*\frac{n}{2}}\right]^{\frac{\gamma}{\gamma-1}} \varphi_0^*$$

$$C^* \frac{y}{\beta_0} = \Sigma_i \frac{2}{2+i} b_i \varphi_0^{*\frac{i}{2}+1} \left[1 + \Sigma_{n \lessgtr 0} \frac{A_n^*}{A_0} \varphi_0^{*\frac{n}{2}}\right]^{\frac{\gamma}{\gamma-1}} + \int_{\varphi_0^*}^{\varphi^*} \Sigma_i b_i \varphi^{*\frac{i}{2}} \left[1 + \Sigma_{n \lessgtr 0} \frac{A_n^*}{A_0} \varphi^{*\frac{n}{2}}\right]^{\frac{\gamma}{\gamma-1}} d\varphi^* .$$

Conversely, if the values of the coefficients $\frac{A_n^*}{A_0}$ $\left(\text{and thus the nature of the pressure distribution } \frac{p_e}{p_\infty}\right)$ are considered to be given as a function of the parameter φ^*, then by means of equations (50), one will be able to determine the corresponding values of the contour generators b_i where one takes into account the determination of b_i for values of $i < 0$ $\left(\text{i.e., for } \varphi^* > \varphi_0^*\right)$, while the values corresponding to $\varphi^* < \varphi_0^*$ are represented by β_w = Constant. In this latter circumstance, therefore, equations (55) constitute parametric relationships for the description of the constraining solid wall corresponding to a given pressure distribution along the edge of the boundary layer, as described by the set of values $\left(\frac{p_e}{p_\infty}, \varphi^*\right)$.

As an intermediate form of proceeding, lying between the two opposite versions mentioned above, one could consider the values of the coefficients $\frac{A_n^*}{A_0}$ to be specified for $n \geqq 0$ and the values of

$b_i = 0$ for $i < 0$. Under this arrangement, then, the relationships spelled out in equations (50) provide the means of obtaining the values of $\frac{A_n^*}{A_0}$ for $n < 0$ together with the values of b_i for $i > 0$, and once these quantities have been elicited, then one may proceed to use equations (55) to determine the shape of the constraining wall.

11. LIMITS OF APPLICABILITY OF THE EXPOUNDED METHOD

The values of φ^* for which the method expounded in the preceding sections is valid are those for which the following inequality holds

$$\frac{A}{A_0} = \Sigma_{n \lessgtr 0} \frac{A_n^*}{A_0} \frac{\varphi^{*\frac{n}{2}}}{\varphi_0^{*\frac{n}{2}}} \varphi_0^{*\frac{n}{2}} < 1$$

and from this condition one may establish, as the lower limit on φ^* for which the method is legitimate, the following criterion:

$$\varphi^*_{min} = \lim_{n \to -\infty} \left(\frac{A_n^*}{A_{n+1}^*} \right)^2 \frac{1}{\lambda} \quad \text{for} \quad n < 0 \quad \text{and for} \quad \lambda < 1$$

and likewise one may set as the upper limit on φ^*, for which application of the method is justified, the following criterion:

$$\varphi^*_{max} = \lim_{n \to +\infty} \left(\frac{A_{n-1}^*}{A_n^*} \right)^2 \lambda \quad \text{for} \quad n > 0 \quad \text{and for} \quad \lambda < 1$$

In both cases one must specify that the sum of the series, which are assuredly going to converge if the stated conditions are obeyed, must be less than unity.

12. DRAG LAW

Once the pressure distribution along the boundary layer has been determined, it then becomes possible to calculate the velocity variations taking place in close proximity to the wall, and with this knowledge at hand, it is then easy to determine the drag. The way these steps may be carried out is as follows:

First let equation (10) be recast into the form

$$\frac{\partial Z}{\partial \varphi^*} = \frac{1}{4}\sqrt{1-\frac{Z}{Z_0}}\,\frac{\partial^2 Z}{\partial \bar{\psi}^{*2}} \qquad [10']$$

and note that in this expression

$$\sqrt{1-\frac{Z}{Z_0}} = \frac{u}{u_e}$$

Now, in order to obtain a description of Z which will be more precise for that region of the boundary layer which is close to the wall than would be obtained by using the solution for the outer region, it may be assumed that in this close-to-the-wall region the velocity-ratio term standing on the right-hand side of equation (10') may be replaced by a relation of the form

$$\frac{u}{u_e} = \mathscr{A}\sqrt{\frac{\bar{\psi}^*}{\bar{\psi}^*_\delta}} \qquad [56]$$

where $\bar{\psi}^*_\delta$ has a value corresponding to the evaluation of θ_δ obtained from equation (39'), where the definition of $\bar{\psi}^*_\delta$ in terms of θ_δ is just $\bar{\psi}^*_\delta = \sqrt{\varphi^*}\,\theta_\delta$.

In this new relationship for the velocity profile near the wall, the constant $\mathscr{A}$ has to be selected in such a way that a reasonably satisfactory approximation will be obtained for the actual well-substantiated law of variation of $\frac{u}{u_e}$ as a function of $\bar{\psi}^*$. Upon examination of profiles put into the form of plots of $\left(\frac{u}{u_e}\,;\,\sqrt{\frac{\bar{\psi}^*}{\bar{\psi}^*_\delta}}\right)$, as obtained by using data for which u_e varies with the x-location by obeying an exponential law (such as treated by Falkner and Skan in their work on the effect of pressure gradients on boundary-layer characteristics) having the form $u_e = K\,x^m$, it is found that $\mathscr{A}$ should take the value 1.3. With this selection for $\mathscr{A}$ it is readily verified that equation (56) gives a good approximation for the way $\frac{u}{u_e}$ varies, regardless of whether the flow is experiencing an acceleration or whether it is going up-hill against an adverse gradient. Thus it may be assumed for present purposes that

$$\sqrt{1-\frac{Z}{Z_0}} = 1{,}3\sqrt{\frac{\bar{\psi}^*}{\psi_\delta^*}}. \qquad [56']$$

Now make the transformation of coordinates defined by

$$\varphi = \int_0^{\varphi^*} \frac{d\varphi^*}{\varphi^{*1/4}\,\theta_\delta^{1/2}} \qquad [57]$$

and thus equation (10') becomes converted to

$$\frac{\partial Z}{\partial \varphi} = \frac{1{,}3}{4}\,\bar{\psi}^{*1/2}\,\frac{\partial^2 Z}{\partial \psi^{*2}} \qquad [58]$$

and if the further change of variable is made that

$$\eta = \sqrt[3]{\frac{16}{1.3}}\,\bar{\psi}^{*1/2} \qquad [59]$$

then the governing differential equation appears as

$$\frac{\partial Z}{\partial \varphi} = \frac{\partial}{\partial \eta}\left(\frac{1}{\eta}\,\frac{\partial Z}{\partial \eta}\right) \qquad [58']$$

and the boundary conditions formerly stated as equation (11) now become

$$Z = 0 \quad \text{for} \quad \eta = \infty \quad ; \quad Z = Z_0 \quad \text{for} \quad \eta = 0 \qquad [11']$$

An appropriate solution of equation (58') which satisfies the boundary conditions stated in equations (11') is

$$Z(\varphi,\eta) = \int_0^{\varphi} \frac{H(\varphi')}{(\varphi-\varphi')^{1/3}}\, e^{-\frac{\eta^3}{9(\varphi-\varphi')}}\, d\varphi' \qquad [60]$$

wherein the function $H(\varphi')$ has to satisfy the integral relationship that

$$Z_0(\varphi) = \int_0^{\varphi} \frac{H(\varphi')\,d\varphi'}{(\varphi-\varphi')^{1/3}} \qquad [61]$$

Thus the H-function may be written as

$$H(\varphi) = \frac{\sin\pi/3}{\pi}\,\frac{d}{d\varphi}\int_0^{\varphi} \frac{Z_0(\varphi')}{(\varphi-\varphi')^{2/3}}\,d\varphi' = \frac{\sin\pi/3}{\pi}\left[\frac{Z_0(0)}{\varphi^{2/3}} + \int_0^{\varphi} \frac{\dot{Z}_0(\varphi')\,d\varphi'}{(\varphi-\varphi')^{2/3}}\right] \qquad [62]$$

where the dot over the Z_0 has the usual significance that

$$\dot{Z}_0(\varphi') = \frac{d Z_0}{d \varphi'} \; .$$

Thus an appropriate solution for Z may be more explicitly written as

$$Z(\varphi, \eta) = \frac{\sin\pi/3}{\pi} \left[Z_0(0) \int_0^{\varphi} \frac{e^{-\frac{\eta^3}{9(\varphi-\varphi')}}}{\varphi'^{2/3}(\varphi-\varphi')^{1/3}} \, d\varphi' + \int_0^{\varphi} \left[\frac{e^{-\frac{\eta^3}{9(\varphi-\varphi')}}}{(\varphi-\varphi')^{1/3}} \right. \right.$$

$$\left. \cdot \int_0^{\varphi'} \frac{\dot{Z}_0(\varphi_1')}{(\varphi-\varphi_1')^{2/3}} \, d\varphi_1' \right] d\varphi' = \frac{\sin\pi/3}{\pi} \left[Z_0(0) \int_0^{\varphi} \frac{e^{-\frac{\eta^3}{9(\varphi-\varphi')}}}{\varphi'^{2/3}(\varphi-\varphi')^{1/3}} \, d\varphi' \right]$$

$$+ \int_0^{\varphi} \dot{Z}_0(\varphi_1') \, d\varphi_1' \int_{\varphi_1'}^{\varphi} \frac{e^{-\frac{\eta^3}{9(\varphi-\varphi')}}}{(\varphi-\varphi')^{1/3}(\varphi'-\varphi_1')^{2/3}} \, d\varphi' .$$

Now the indicated integrals appearing here may be evaluated to give the following results:

$$\int_0^{\varphi} \frac{e^{\frac{\eta^3}{9(\varphi-\varphi')}}}{\varphi'^{2/3}(\varphi-\varphi')^{1/3}} \, d\varphi' = \Gamma\left(\frac{1}{3}\right)\left[\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{\eta^3}{9\varphi} ; \frac{2}{3}\right)\right]$$

$$\int_{\varphi_1'}^{\varphi} \frac{e^{-\frac{\eta^3}{9(\varphi-\varphi')}}}{(\varphi-\varphi')^{1/3}(\varphi'-\varphi_1')^{2/3}} \, d\varphi' = \Gamma\left(\frac{1}{3}\right)\left\{\Gamma\left(\frac{2}{3}\right) - \Gamma\left[\frac{\eta^3}{9(\varphi-\varphi_1')} ; \frac{2}{3}\right]\right\}$$

wherein the symbol $\Gamma(\theta, n)$ is used to denote the incomplete gamma function, defined as

$$\Gamma(\theta, n) = \int_0^{\theta} t^{n-1} e^{-t} \, dt \; .$$

Consequently, the sought solution for Z may now be written as

$$Z(\varphi, \eta) = \frac{\sin\pi/3}{\pi} \Big[Z_0(0) \, \Gamma\left(\frac{1}{3}\right) \Big\{ \Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{\eta^3}{9\varphi} ; \frac{2}{3}\right) \Big] - \int_0^{\varphi} \dot{Z}_0(\varphi_1') \, \Gamma\left(\frac{1}{3}\right) \Big\{ \Gamma\left(\frac{2}{3}\right)$$

$$- \Gamma\left[\frac{\eta^3}{9(\varphi-\varphi_1')} ; \frac{2}{3}\right] \Big\} d\varphi_1' = \frac{\sin\pi/3}{\pi} \, \Gamma\left(\frac{1}{3}\right) \Big\{ - Z_0(0) \, \Gamma\left(\frac{\eta^3}{9\varphi} ; \frac{2}{3}\right) +$$

$$+ Z_0(\varphi) \, \Gamma\left(\frac{2}{3}\right) - \int_0^{\varphi} \dot{Z}_0(\varphi_1') \, \Gamma\left[\frac{\eta^3}{9(\varphi-\varphi_1')} ; \frac{2}{3}\right] d\varphi_1' \Big\}$$

which gives the velocity distribution occurring in the boundary layer in proximity to the constraining wall.

Next, then, it will be of interest to find the value of $\left(\frac{\partial Z}{\partial \overline{\psi}^*}\right)_{\overline{\psi}^*=0}$,
inasmuch as the skin-friction drag may be evaluated once this derivative is known, because the velocity gradient is related to this derivative in the following way:

$$\frac{\partial Z}{\partial \psi^*} = -\frac{2u}{1-u_e^2}\frac{\partial u}{\partial \psi^*} = -\frac{2u_\infty}{1-u_e^2}\frac{\rho_\infty}{\rho}\frac{\partial u}{\partial y}.$$

It is immediately apparent what the value of the derivative is by reference to the expression for Z given above, and when evaluated at $\overline{\psi}^* = 0$, the result is

$$\left(\frac{\partial Z}{\partial \overline{\psi}^*}\right)_{\overline{\psi}^*=0} = \frac{\sin\pi/3}{\pi}\left[-Z_0(0)\,\Gamma\left(\frac{1}{3}\right)\frac{9^{1/3}}{6}\left(\frac{16}{1.3}\right)^{2/3}\frac{1}{\varphi^{2/3}} - \Gamma\left(\frac{1}{3}\right)\frac{9^{1/3}}{6}\left(\frac{16}{1.3}\right)^{2/3} \cdot \int_0^\varphi \dot{Z}_0(\varphi_1')\frac{d\varphi_1'}{(\varphi-\varphi_1')^{2/3}}\right] = -\frac{\sin\pi/3}{\pi}\,\Gamma\left(\frac{1}{3}\right)\frac{9^{1/3}}{6}\left(\frac{16}{1.3}\right)^{2/3}\int_0^\varphi\frac{dZ_0(\varphi_1')}{(\varphi-\varphi_1')^{2/3}}$$

The frictional shear stress may thus be evaluated by use of the information at hand, inasmuch as

$$\tau_w = \mu_w\left(\frac{\partial U}{\partial Y}\right)_{Y=0} = \frac{\mu_w U_l}{L}\left(\frac{\partial u}{\partial y}\right)_{y=0} = -C_w\frac{\mu_\infty U_\infty}{L}\frac{1-u_e^2}{2u_\infty^2}\frac{\sqrt{Re}}{2}\left(\frac{\partial Z}{\partial \overline{\psi}^*}\right)_{\overline{\psi}^*=0}$$

where an affine connection between wall values and free-stream values for the product $\mu\rho$ has been assumed, that is, where use has been made of the substitution that $\mu_w\rho_w = C_w\rho_\infty\mu_\infty$. It should be observed that in this system of notation $C_w = C^*$.

Thus, because it has been pointed out earlier that $\frac{u_\infty^2}{(1-u_e^2)} \cong A_0$, it follows that the sought expression for the skin-friction coefficient may be written as

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{C_w}{\sqrt{Re}}\frac{\sin\pi/3}{\pi}\Gamma\left(\frac{1}{3}\right)\frac{9^{1/3}}{12}\left(\frac{16}{1.3}\right)^{2/3}\int_0^\varphi\frac{dZ_0/A_0}{(\varphi-\varphi_1')^{2/3}} = 0.6826\frac{C_w}{\sqrt{Re}}\int_0^\varphi\frac{dZ_0/A_0}{(\varphi-\varphi_1')^{2/3}}\cdot \quad [63]$$

13. EVALUATION OF THE INTEGRAL APPEARING IN THE SKIN-FRICTION FORMULA AND SUBSTANTIATION OF A SIMPLIFIED EXPRESSION FOR C

In order to carry out the indicated integration which is required for evaluation of the skin-friction coefficient, as just defined above, it will be found especially convenient to proceed by making the substitution

$$\varphi = a\ \varphi^{*3/4} \qquad [64$$

in the expression in question, namely, $\int_0^{\varphi} \frac{dZ_0/A_0}{(\varphi - \varphi_1')^{2/3}}$ where it is assumed that a is a constant.

This assumption for the affine relation between φ and $\varphi^{*3/4}$ is justified on the basis of the results of detailed calculations such as those made subsequently in sections 14 and 15. In fact, it is seen from these calculated results that the ratio $\frac{\varphi}{\varphi^{*3/4}}$ is actually always going to remain very close to unity for a very remarkably large range of variation in the parameters χ_e and χ_e'.

Now, by changing the limits of integration, one has that

$$\int_0^{\varphi} \frac{dZ_0/A_0}{(\varphi - \varphi'_1)^{2/3}} = \frac{1}{A_0}\,\frac{Z_0(\varphi_0^*)}{\varphi^{2/3}} + \frac{1}{A_0}\int_0^{\varphi} \frac{dZ_0}{d\varphi'_1}\,\frac{d\varphi'_1}{(\varphi - \varphi'_1)^{2/3}}$$

where φ_0 is used to denote the value of φ corresponding to $\varphi^* = \varphi_0^*$.

In addition, previous analysis gives that

$$\frac{1}{A_0}\,\frac{dZ_0}{d\varphi^*} = \Sigma_{n \geq 0}\,\frac{n}{2}\,\frac{A_n^*}{A_0}\,\varphi^{*\frac{n}{2}-1}$$

and thus, thanks to the affine relation specified in equation (64), the evaluation of the integral appearing in the expression for the skin friction may be reduced merely to the requirement for evaluation of the series

$$I = \frac{4}{3}\,\Sigma_{n \geq 0}\, a^{-\frac{2}{3}n}\,\frac{n}{2}\,\frac{A_n^*}{A_0}\int_{\varphi_0}^{\varphi} \varphi_1^{\frac{2n}{3}-1}\,(\varphi - \varphi_1)^{-2/3}\,d\varphi_1. \qquad [65]$$

From this starting point, then, in the case where $n > 0$, it is found by replacing $\frac{\varphi_1}{\varphi}$ by t that

$$\int_{\varphi_0}^{\varphi} \varphi_1^{\frac{2n}{3}-1} (\varphi-\varphi_1)^{-\frac{2}{3}} d\varphi_1 \cong \int_{0}^{\varphi} \varphi_1^{\frac{2n}{3}-1} (\varphi-\varphi_1)^{-\frac{2}{3}} d\varphi_1 = \int_0^1 \varphi^{p+q+1} t^{p-1} (1-t)^{q-1} dt$$
$$= \varphi^{p+q+1} B(p,q) = \varphi^{p+q+1} \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}$$

where $p = \frac{2n}{3}$ and $q = \frac{1}{3}$.

In the case where $n < 0$ there is no need to consider other than the following three discrete cases:

(a) For $n = -1$ the integral becomes

$$\int_{\varphi_0}^{\varphi} \varphi_1^{-2/3-1} (\varphi-\varphi_1)^{-2/3} d\varphi_1 \cong \varphi^{-4/3} \Big[1.5\left(\frac{\varphi_0}{\varphi}\right)^{-2/3} - 2\left(\frac{\varphi_0}{\varphi}\right)^{1/3} - 0.41666\left(\frac{\varphi_0}{\varphi}\right)^{4/3}$$
$$- 0,21164\left(\frac{\varphi_0}{\varphi}\right)^{7/3} - 0,135802\left(\frac{\varphi_0}{\varphi}\right)^{10/3} - 0,097499\left(\frac{\varphi_0}{\varphi}\right)^{13/3} - 0,074817\left(\frac{\varphi_0}{\varphi}\right)^{16/3} + 2.64912\Big]$$

provided it is taken for granted that $\frac{\varphi_0}{\varphi} \leqq 0.5$.

(b) For $n = -2$ the integral becomes

$$\int_{\varphi_0}^{\varphi} \varphi_1^{-4/3-1} (\varphi-\varphi_1)^{-2/3} d\varphi_1 = \frac{3}{\varphi^2}\left[\frac{1}{4}\left(\frac{\varphi}{\varphi_0}-1\right)^{4/3} + \left(\frac{\varphi}{\varphi_0}-1\right)^{1/3}\right]$$
$$= \frac{3}{\varphi^2}\left(\frac{\varphi}{\varphi_0}-1\right)^{1/3}\left[\frac{1}{4}\left(\frac{\varphi}{\varphi_0}-1\right)+1\right] = \frac{3}{\varphi^2}\left(\frac{\varphi}{\varphi_0}-1\right)^{1/3}\left(\frac{1}{4}\frac{\varphi}{\varphi_0}+\frac{3}{4}\right) = \frac{3}{4\varphi^2}\left(\frac{\varphi}{\varphi_0}-1\right)^{1/3}\left(\frac{\varphi}{\varphi_0}+3\right)$$

(c) For $n = -3$ the integral becomes

$$\int_{\varphi_0}^{\varphi} \varphi_1^{-3} (\varphi-\varphi_1)^{-2/3} d\varphi_1 = -\frac{1}{6\varphi^2}\left(\frac{3\varphi t}{\varphi_0^2} + \frac{5t}{3\varphi_0} + 10J\right) - 0,503833\,\varphi^{-8/3}$$

where $t = -\sqrt[3]{\varphi - \varphi_0}$

$$J = \frac{1}{6\varphi^{2/3}}\left[\log\frac{(t+\varphi^{1/3})^2}{t^2-\varphi^{1/3}t+\varphi^{2/3}} + 2\sqrt{3}\,\tan^{-1}\frac{2t-\varphi^{1/3}}{\varphi^{1/3}\sqrt{3}}\right]$$

Having carried out these detailed evaluations for I, the following pertinent summary formula for calculation of the skin-friction coefficient may be used:

$$\sqrt{Re}\,\bar{\varphi}^{*1/2}\,C_f = \sqrt{C_w}\,0.6826\,[a^{-3/4} + a^{1/4}\varphi^{*1/2} I] \qquad [66]$$

At this point it is worth noting that if one drops out of this expression any contribution to the skin friction arising because of the presence of the self-induced pressure gradients, and if just the case of a flow over a flat plate is considered, then the above-written expression reduces to merely

$$\sqrt{R_x}\,C_f = \sqrt{C_w}\,0.6826 \qquad [66']$$

while it will be recalled that the widely used Rubesin and Chapman result (ref. 6) for such uncomplicated conditions is

$$\sqrt{R_x}\,C_f = \sqrt{C_w}\,0.664$$

Consequently, it appears that the skin-friction coefficient C_f is given by the present analysis to an accuracy which deviates only about 1.8 percent from the well-established value that has been found to be generally applicable.

For purposes of a further check on the present derivations, comparison can also be made with the particular pressure-gradient case studied by Falkner and Skan for which $\frac{u_e}{u_\infty} = hx$ in an incompressible stream. In terms of the developments adduced above, the case in question is handled by first evaluating φ as

$$\varphi = \int_0^{\varphi^*} \frac{d\varphi^*}{\theta_\delta^{1/3}\,\varphi^{*1/4}} = \frac{4}{3}\,\frac{1}{\theta_\delta^{1/3}}\,\varphi^{*3/4}$$

inasmuch as under present circumstances, as is demonstrated in appendix C, the value of θ_δ is a constant and, in fact, $\theta_\delta = 1.46$.

Meanwhile, it is also true for the present case that

$$\frac{1}{u_\infty^2}\,\frac{dZ_0}{d\varphi} = h\left(\frac{3}{4}\right)^{1/3}\theta_\delta^{1/3}\,\varphi^{1/3}$$

and, consequently, the skin-friction coefficient may be evaluated from the relation

$$C_f = \frac{1}{\sqrt{Re}}\,\frac{\sin\pi/3}{\pi}\,\Gamma\left(\frac{1}{3}\right)\left(\frac{16}{1.3}\right)^{2/3}\frac{9^{1/3}}{12}\,h\left(\frac{3}{4}\right)^{1/3}\theta_\delta^{1/3}\int_0^{\varphi}\frac{\varphi_1^{1/3}\,d\varphi_1}{(\varphi-\varphi_1)^{2/3}}$$

but because the indicated integral may be evaluated as

$$\int_0^{\varphi} \frac{\varphi_1^{1/3}\, d\,\varphi_1}{(\varphi-\varphi_1)^{2/3}} = \varphi^{2/3}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$$

it follows that, for this special type of pressure gradient, the present method predicts that

$$C_f = 2.187\,\frac{h^{1/3}\,x}{\sqrt{Re}}$$

It may be recalled that the equivalent expression obtained by Hartree in this same case is, instead,

$$C_f = 2.46\,\frac{h^{1/3}\,x}{\sqrt{Re}}$$

and, consequently, it may be seen that the error involved in using equation (66) for the case under consideration is only about 11.4 percent from the accepted value.

On the basis of these checks it appears, therefore, that the expression for the skin-friction coefficient, as embodied in equation (66), will be sufficiently accurate for most purposes.

14. APPLICATIONS

A numerical application of the theoretical method expounded above will now be carried out for the following two cases of interest:

(a) a curved constraining wall defined by the relation

$$\beta_w = \frac{d\,y}{d\,x} = \beta_0\,\Sigma_i\, b_i\, \varphi^{*\frac{i}{2}} \qquad [67]$$

in which

$$\beta_0 = 0.087 \;;\; b_0 = 1 \;;\; b_2 = -2 \;;\; b_4 = -0.98 \;;\; b_i = 0 \text{ for } i \neq \begin{cases}0\\2\\4\end{cases}$$

(b) a flat plate for which $\beta_w = 0$.

For both of these conditions the calculations are to be made for the following six selections of concrete values for the Mach number and the Reynolds number: Reynolds numbers taken as $Re = 10^6$ and 10^7 and Mach numbers taken as $M_\infty = 5, 6,$ and 8.

A typical value for the total temperature is selected as $T_\infty = 250^\circ$ K. On the basis of this assumption the value of the constant C^* may be obtained from the relation

$$C^* = \frac{\mu_w T_\infty}{\mu_\infty T_w} = \sqrt{\frac{T_w}{T_\infty}}\,\frac{T_\infty + S}{T_w + S} = \sqrt{\frac{T_w}{T_\infty}}\,\frac{T_\infty + 120}{T_w + 120}$$

where the relation between viscosity and temperature is assumed to follow the Sutherland law. Now the $\frac{T_w}{T_\infty}$ ratio is obtained from

$$\frac{T_w}{T_\infty} = 1 + \frac{\gamma - 1}{2} M_\infty^2$$

and consequently,

$$C^* = C_w = \begin{cases} 0.55945 & \text{for } M_\infty = 5.0 \\ 0.48826 & \text{for } M_\infty = 6.0 \\ 0.38501 & \text{for } M_\infty = 8.0 \end{cases}$$

The corresponding values for the parameters χ_e and χ_e' are summarized in table I.

The values of the constants K_0, $K^*_{1,m}$, $K^*_{1,m,n}$, which appear in equations (47), (48), and so forth, have been calculated once and for all by assuming for φ_0 the value 4/500, which may be considered as a guessed trial value, but which really does not have much of an influence on the final result and will not be altered in succeeding steps. These K-values have been collected to form table II.

By means of the determining relationships set down as equations (50), (51), and so forth, the values of the coefficients $\frac{A_n^*}{A_0}$ have been ascertained and the results are presented in tabular form. These values, presented as table III, pertain to the case of the curved constraining wall.

By use of equation (67) the x- and y-coordinates of the curved constraining wall have then been computed from the values given in table III and the results are presented in table IV.

Graphs have been constructed which show the shape of the airfoils corresponding to the coordinates given in the tables, and these contours are given with 5 times magnification of the vertical scale in figures 1 and 2.

The corresponding pressure distributions $\frac{p_e}{p_\infty}$ have been calculated at the points along the airfoil contours by making use of the values of $\frac{A_n^*}{A_0}$ which have been presented in table III. The location parameter for the pressures is φ^*, and the pertinent sets of $\frac{p_e}{p_\infty}$ values are displayed in table V.

The pressure distributions which are produced on the airfoil shapes displayed in figures 1 and 2 are also depicted in the upper set of graphs in figures 1 and 2, by plotting the data adduced in table V.

In order to arrive at an evaluation of the skin friction produced on such airfoils, it is first necessary to obtain values of θ_δ which hold at the outer edge of the boundary layer. These values may be obtained by solving equation (39'). For convenience it may be assumed that $C_0 = 0.925^2 = 0.990025$, and then by making use of the data of table III, the sought values of θ_δ may be determined for the usual sequence of φ^* values. These operations have been carried out and the results are collected into table VI.

The values of φ which correspond to the selected values of φ^* for the various Mach number and Reynolds number combinations now under consideration are presented in table VII. It may be observed from scrutiny of these values that the supposition that $\varphi = a\varphi^{*3/4}$ (as suggested in eq. (64), with a assigned the value unity) is quite amply justified. Moreover it will be seen from the results that setting $a = 1$ is legitimate in every single case for the whole range of variables under consideration; as an example, even for the condition of $M_\infty = 8$ and $Re = 10^6$, the proportionality constant between $\varphi^{*3/4}$ and φ is actually 1.041, which is close enough to unity for practical purposes.

Making use of the tabular information that has already been referred to, one may now proceed easily, through use of equation (66), to find the skin-friction coefficient C_f which will be given in the combined form of a term denoted by $\sqrt{Re}\,\overline{\varphi}^{*1/2}C_f$. These values are presented in table VIII. They have also been plotted according to their location along the airfoil chord, in the upper graphs of figures 1 and 2. In order to plot against the X/L-coordinate, one needs to use the data from table VIII in conjunction with the information given in table IV.

It may be seen from examination of the profile contours given in figure 2, which apply for $Re = 10^7$, that the contour shapes are practically indistinguishable from one another for the three Mach numbers illustrated. On the other hand, the graphs showing the corresponding pressure distributions for these three profiles exhibit a rather marked difference, and this notable variation takes place even though the highest value of X_e that is involved is only $X_e = 0.124$. When one looks at the similar curves for the pressure distributions that have been obtained for the other three profiles, pertaining to the lower Reynolds number situation, for which $Re = 10^6$, the differences between the distributions are even more accentuated than those for the $Re = 10^7$ case, but this significant spread in the distributions in this latter case ($Re = 10^6$) is to be expected because of the more elevated values of X_e that are involved. The pressure differences that are observed in this latter case are, in fact, not surprising, because the corresponding contour shapes are appreciably different, as shown in figure 1, even though the constants b_i are the same, of course, in all three cases.

15. THE FLAT PLATE - NUMERICAL APPLICATION

By returning once again to use of equations (50), (51), and so forth, the values of the coefficients $\frac{A_n^*}{A_0}$ have been determined in the present case of interest, where the profile contour has been assumed to be that of a flat plate and where the same flight conditions of Mach number M_∞ and Reynolds number Re, as used in the preceding section, are again selected for illustration. The computed values of $\frac{A_n^*}{A_0}$ for these conditions are given in table IX. These coefficients have been used to compute the relation between φ^* and the x-coordinate on the flat plate, and the results are presented in table X.

In order to arrive at an evaluation for the skin friction, the necessary task of calculating θ_δ has been carried out for the case of the flat plate under consideration, and the resulting values are adduced in table XI.

When the values of φ are computed, which correspond to the values of φ^* that have been used here, it turns out that the constant a appearing in equation (64) takes on an average value which is very close to all the individual evaluations of the ratio $\frac{\varphi}{\varphi^{*3/4}}$. For the several different flight conditions under consideration, it is found that these various median values for a are the following:

$$M_\infty = 5 \begin{cases} R_e = 10^6 ; a = 0.98814 ; \\ R_e = 10^7 ; a = 0.98821 ; \end{cases} \quad M_\infty = 6 \begin{cases} R_e = 10^6 ; a = 0.98857 ; \\ R_e = 10^7 ; a = 0.98852 ; \end{cases} \quad M_\infty = 8 \begin{cases} R_e = 10^6 ; a = 0.97213 \cdot \\ R_e = 10^7 ; a = 0.98808 . \end{cases}$$

Making use of the information that has now been assembled, in this case of flat-plate flow, the pressure distributions and reduced skin-friction coefficients $A = \sqrt{Re}\sqrt{x}\, C_f$, which are locally realized along the flat plate, have been computed and the numerical data are given in table XII, while the graphical display of these flat-plate airfoil boundary-layer characteristics are given in figures 3 and 4.

The influence of the parameter χ_e in producing significantly different levels in the pressure distributions is again evident from these flat-plate data. It is especially worthy of note that the data show a trend of increasing skin-friction coefficient with increasing Mach number for the larger χ_e values. This behavior is in direct contrast with what is seen to take place at values near $\chi_e = 0$. This astonishing result thus runs counter to what is known to occur in reality and to be predicted theoretically at lower supersonic Mach numbers.

16. COMPARISON OF THE THEORY WITH EXPERIMENTAL RESULTS

In order to give a check on the validity of the deductions and formulas presented in this analysis, a comparison can be made between these theoretical predictions and the results found from experiment, such as those reported by M. H. Bertram in NACA TN 2773 (ref. 7), which pertain to the boundary layer on a flat plate immersed in a flow with hypersonic speed of $M_\infty = 6.86$ and with a Reynolds number for the section of $Re = 0.98 \times 10^6$.

The confrontation of theory and experiment in this case will be made with respect to the pressure distributions which were measured, as compared to those predicted here. The theoretical data points for the Mach number of interest $M_\infty = 6.86$ have been obtained by interpolating between the calculated values which apply for $M_\infty = 6$ and $M_\infty = 8$, at a Reynolds number of 10^6. The theoretical curve, thus obtained, has been sketched in on figure 3 by means of a dashed line. Even though the scatter in the experimental points seems to be excessive, nevertheless, the theoretical curve, devised as just stated, appears to constitute a very good mean line for the experimental findings, particularly for values of X/L greater than 0.15.

APPENDIX A

TEMPERATURE-DENSITY RELATIONSHIP EMPLOYED IN THE MOMENTUM EQUATION

In order to arrive at the simple form given in equation (10) for the differential equation governing the flow in the boundary layer, it is necessary to make use of the relation $\frac{T}{T_e} = \frac{\rho_e}{\rho}$; this simplification may be substantiated in the following way.

When the terms representing the effect of viscosity are dropped out of the momentum equation describing the flow in the boundary layer, then the resulting more wieldy version may be written as

$$\frac{\partial}{\partial \psi^*}\left(\frac{p}{p_\infty}\right) = -\gamma M_\infty^2 \frac{\partial}{\partial x}\left(\beta \frac{u}{u_\infty}\right) \qquad [A\text{-}1]$$

where the new variables have been introduced through recourse to the Von Mises transformation, as employed to obtain equation (5) of the main text.

The formal solution for the pressure difference existing at the wall and at the outer edge of the boundary layer may thus be seen to be given by

$$\frac{p_e}{p_\infty} - \frac{p_w}{p_\infty} = -\gamma M_\infty^2 \int_0^{\psi_\delta^*} \frac{\partial}{\partial x}\left(\beta \frac{u}{u_\infty}\right) d\psi^* = -\gamma M_\infty^2 \frac{\rho_e}{\rho_\infty} \frac{u_e}{u_\infty} C^* \int_0^{\psi_\delta^*} \frac{\partial}{\partial \varphi^*}\left(\beta \frac{u}{u_\infty}\right) d\psi^*$$

Consequently,

$$\frac{p_w}{p_\infty} = 1 + \gamma M_\infty^2 \frac{u_e}{u_\infty} C^* \int_0^{\psi_\delta^*} \frac{\partial}{\partial \varphi^*}\left(\beta \frac{u}{u_\infty}\right) d\psi^* \cong 1 + \frac{\gamma M_\infty^2 C^2 \sqrt{\varphi^*}}{\sqrt{Re}} \int_0^{\theta_\delta} \frac{\partial}{\partial \varphi^*}\left(\beta \frac{u}{u_e}\right) d\theta .$$

Now, inasmuch as it was previously agreed that

$$\frac{u}{u_e} = \sqrt{\frac{Z_0 - Z}{Z_0}} \; ; \; \beta = \beta_w + \frac{1}{\sqrt{R_e}} \frac{p_e}{p_\infty} \sqrt{\varphi^*} \int_0^\theta \frac{\partial}{\partial \varphi^*}\left(\frac{\rho_\infty}{\rho} \frac{u_\infty}{u}\right) d\theta$$

$$= \frac{\chi'_e}{M_\infty} \Sigma_i b_i \varphi^{*\frac{i}{2}} + \frac{\gamma - 1}{2} \frac{\chi_e}{M_\infty} \sqrt{\varphi^*} \frac{p_e}{p_\infty} \int_0^\theta \frac{\partial}{\partial \varphi^*}\left(\frac{\rho_\infty}{A_0 \rho} \frac{u_\infty}{u}\right) d\theta$$

it follows that

$$\frac{p_w}{p_\infty} = 1 + \gamma C^* \sqrt{\varphi^*} \left\{ \frac{\chi_* \chi'_*}{M_\infty^2} \int_0^{\theta_\delta} \frac{\partial}{\partial \varphi^*} \left(\frac{u}{u_*} \Sigma_i b_i \varphi^{*\frac{1}{2}} \right) d\theta \right.$$

$$\left. + \frac{\gamma - 1}{2} \frac{\chi_*^2}{M_\infty^2} \int_0^{\theta_\delta} \frac{\partial}{\partial \varphi^*} \left[\frac{p_*}{p_\infty} \sqrt{\varphi^*} \int_0^{\theta} \frac{\partial}{\partial \varphi^*} \left(\frac{\rho_\infty}{A_0 \rho} \frac{u_\infty}{u} \right) d\theta \right] d\theta \right\} = 1 + O\left(\frac{\chi_* \chi'_*}{M_\infty^2} \right) + O\left(\frac{\chi_*^2}{M_\infty^2} \right) \cong 1 .$$

Thus it appears to be entirely permissible to use the relation

$$\frac{\rho_*}{\rho} = \frac{p_*}{p} \frac{T}{T_*} \cong \frac{T}{T_*}$$

in the first momentum equation of the main text.

APPENDIX B

DETAILED DETERMINATION OF THE $F_n^*(\theta)$ FUNCTIONS APPEARING IN EQUATION (19) OF THE TEXT

In the development of equation (19) from equation (17) of the main text, it was admitted that certain approximations were being indulged in; the step-by-step evaluation of the pertinent integrals involved is given below, in order to show the precise place of introduction and the slight degree of importance of the simplifying tactics employed in arriving at the description of the $F_n^*(\theta)$ functions appearing in equation (19) of the text.

The integral to be approximated is, for the general case,

$$\frac{\psi^*}{\sqrt{\pi}}\int_0^{\varphi^*}\frac{e^{-\frac{\overline{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^*-\varphi')^{3/2}}A(\varphi')d\varphi' \quad \text{for} \quad \begin{cases} A(\varphi) = \sum\limits_{n=-k}^{-1} A_n^*\varphi^{*n/2} & \text{when} \quad \varphi^* > \varphi_0^* \\ A(\varphi) = \sum\limits_{n=-k}^{-1} A_n^*\varphi_0^{*n/2} & \text{when} \quad \varphi^* < \varphi_0^* \end{cases}$$

Consider the following specific case for the first of the sequence of n-values:

In this first case, where $n = -1$, an approximation for the sought evaluation can be obtained by proceeding in the following way. Consider the integral

$$\frac{\overline{\psi}^*}{\sqrt{\pi}}\int_{\varphi_0^*}^{\varphi^*}\frac{e^{-\frac{\overline{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^*-\varphi')^{3/2}}\frac{d\varphi'}{\varphi'^{1/2}} = \frac{\overline{\psi}^*}{\sqrt{\pi}}\int_0^{\varphi^*}\frac{e^{-\frac{\overline{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^*-\varphi')^{3/2}}\frac{d\varphi'}{\varphi'^{1/2}}$$

$$-\frac{\overline{\psi}^*}{\sqrt{\pi}}\int_0^{\varphi_0^*}\frac{e^{-\frac{\overline{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^*-\varphi')^{3/2}}\frac{d\varphi'}{\varphi'^{3/2}}$$

from which it is readily perceived that

$$\frac{\bar{\psi}^*}{\sqrt{\pi}} \int_{\varphi_0^*}^{\varphi^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^* - \varphi')^{3/2}} \frac{d\varphi'}{\varphi'^{1/2}} = \frac{1}{\sqrt{\varphi^*}} e^{-\frac{\bar{\psi}^{*2}}{\varphi^*}} - \frac{\bar{\psi}^*}{\sqrt{\pi}} \int_0^{\varphi_0^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^* - \varphi')^{3/2}} \frac{d\varphi'}{\varphi'^{3/2}}$$

Now let the simplification in notation be made that $\theta = \frac{\bar{\psi}^*}{\sqrt{\varphi^*}}$, and thus the indicated evaluation for $\varphi^* \gg \varphi_0^*$ is obtainable as

$$\frac{\bar{\psi}^*}{\sqrt{\pi}} \int_0^{\varphi_0^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^* - \varphi')^{3/2}} \frac{d\varphi'}{\varphi'^{1/2}} = \frac{2}{\pi} \int_{\frac{\bar{\psi}^*}{\varphi^{*1/2}}}^{\frac{\bar{\psi}^*}{\varphi^{*1/2}} \frac{1}{\left(1-\frac{\varphi_0^*}{\varphi^*}\right)^{1/2}}} \frac{e^{-\beta^2} d\beta}{\varphi^{*1/2}\left(1-\frac{\theta^2}{\beta^2}\right)^{1/2}} = \frac{2}{\sqrt{\pi}} \frac{1}{\varphi^{*1/2}} \int_0^{\theta\left(1-\frac{\varphi_0^*}{\varphi^*}\right)^{-1/2}} \frac{e^{-\beta^2} d\beta}{\left(1-\frac{\theta^2}{\beta^2}\right)^{1/2}}$$

$$\cong \frac{2}{\sqrt{\pi}} \frac{1}{\varphi^{*1/2}} e^{-\theta^2} \theta^2 \frac{\varphi_0^*}{\varphi^*}$$

Consequently, the evaluation of the first integral under consideration is obtained from the following development:

$$I_{-1} = A^*_{-1} \frac{\bar{\psi}^*}{\sqrt{\pi}} \int_{\varphi_0^*}^{\varphi^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^* - \varphi')^{3/2}} \frac{d\varphi'}{\varphi'^{1/2}} + A^*_{-1} \varphi_0^{*-1/2} \frac{\bar{\psi}^*}{\sqrt{\pi}} \int_0^{\varphi_0^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^*-\varphi'}}}{(\varphi^* - \varphi')^{3/2}} d\varphi'$$

$$= A^*_{-1} \left[\frac{1}{\sqrt{\varphi^*}} e^{-\frac{\bar{\psi}^{*2}}{\varphi^*}} - \frac{2}{\sqrt{\pi}} \frac{\varphi_0^*}{\varphi^{*3/2}} \theta^2 e^{-\theta^2}\right] + A^*_{-1} \varphi_0^{*-1/2} \left\{- erfc\left[\frac{\theta}{\left(1-\frac{\varphi_0^*}{\varphi^*}\right)^{1/2}}\right] + erfc\, \theta\right\} .$$

The quantity set off within brackets reduces simply to

$$- erfc\left[\frac{\theta}{\left(1-\frac{\varphi_0^*}{\varphi^*}\right)^{1/2}}\right] + erfc\, \theta \cong - erfc\left[\theta\left(1+\frac{1}{2}\frac{\varphi_0^*}{\varphi^*}\right)\right] - erfc\, \theta \cong \frac{e^{-\theta^2}}{2\sqrt{\pi}} \theta \frac{\varphi_0^*}{\varphi^*} .$$

Consequently, it is evident that the first integral of concern may be evaluated for $\varphi^* \gg \varphi_0^*$ as

$$I_{-1} = A^*_{-1} \frac{e^{-\theta^2}}{\varphi^{*1/2}} .$$

In the case where $n = -2$, consider the :ntegral

$$\frac{\bar{\psi}^*}{\sqrt{\pi}} \int_{\varphi_0^*}^{\varphi^*} \frac{e^{-\frac{\bar{\psi}^{*2}}{\varphi^* - \varphi'}}}{(\varphi^* - \varphi')^{1/2}} \frac{d\varphi'}{\varphi'} = \frac{2}{\sqrt{\pi}} \frac{1}{\varphi^*} \int_{\theta\left(1 - \frac{\varphi_0^*}{\varphi^*}\right)^{1/2}}^{\infty} \frac{e^{-\beta^2} d\beta}{1 - \theta^2/\beta^2}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{\varphi^*} \left[\theta^2 \int_{\theta\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2}}^{\infty} \frac{e^{-\beta^2}}{\beta^2 - \theta^2} d\beta + \int_{\theta\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2}}^{\infty} e^{-\beta^2} d\beta \right] .$$

Now let the first of these above-written integrals be more closely examined. It will be seen that

$$\theta^2 \int_{\theta\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2}}^{\infty} \frac{e^{-\beta^2}}{\beta^2 - \theta^2} d\beta = \theta^2 \int_{\theta\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2}}^{\infty} \frac{e^{-\beta^2} - e^{-\theta^2}}{\beta^2 - \theta^2} d\beta + \theta^2 e^{-\theta^2} \int_{\theta\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2}}^{\infty} \frac{d\beta}{\beta^2 - \theta^2}$$

and it is further readily recognized that the first of these two contributions to the right-hand side will remain finite as $\varphi_0^* \to 0$, and, consequently, it remains finite for $\varphi^* \gg \varphi_0^*$. Thus it is permissible to replace the lower limit in this first integral by the value θ; then one may write it as

$$\theta^2 \int_{\theta}^{\infty} \frac{e^{-\beta^2} - e^{-\theta^2}}{\beta^2 - \theta^2} d\beta = \theta^2 H(\theta)$$

When this is done, it may be observed that $H(\theta) \to 0$ as $\theta \to \infty$.

Now consider the other contributing part of the right-hand side of the above equation; in this instance it may be observed that

$$e^{-\theta^2}\theta^2 \int_{\theta\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2}}^{\infty} \frac{d\beta}{\beta^2 - \theta^2} = -\frac{\theta}{2} e^{-\theta^2} \log \frac{\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2} - 1}{\left(\frac{\varphi^*}{\varphi^* - \varphi_0^*}\right)^{1/2} + 1} \cong -e^{-\theta^2} \frac{\theta}{2} \log \frac{\varphi_0^*}{4\varphi^*}$$

Then of course, for $1 > \varphi^* \gg \varphi_0^*$, it is true that

$$\left| \log \frac{\varphi_0^*}{4} \right| \gg \left| \log \varphi^* \right|$$

so that, finally, one may give the approximate evaluation for the second term in the series expansion under examination for $\varphi^* \gg \varphi_0^*$ as

$$I_{-2} = A^*_{-2} \frac{1}{\varphi^*} \left[erfc(\theta) - \frac{\theta}{\sqrt{\pi}} e^{-\theta^2} \log \frac{\varphi_0^*}{4} \right] .$$

This approximation that has now been made in arriving at this result for I_{-2} is not really essential to the development being expounded. In fact, rather than to follow the recommended procedure, as previously given, by ignoring in the expression for I_{-2} the term involving $\frac{1}{\varphi^*}\log\varphi^*$ $\left(\text{in comparison with the term } \frac{1}{\varphi^*}\log\frac{\varphi_0^*}{4}\right)$, this term could easily be taken into account. To do this, while not changing the expression assumed for $A(\varphi)$, one would be led to take additional adjusting terms in the development for β_W, other than those prescribed in the treatment given in the main text. These additional terms would adequately compensate for the presence of the previously ignored term $\frac{1}{\varphi^*}\log\varphi^*$.

Continuing on with the evaluations, the following manipulations are found appropriate:

In this case where $n = -3$, one has to assess the value of

$$\frac{\bar\psi^*}{\sqrt{\pi}}\int_{\varphi_0^*}^{\varphi^*}\frac{e^{-\frac{\bar\varphi^{*2}}{\varphi^*-\varphi'}}}{(\varphi^*-\varphi')^{3/2}}\,\frac{1}{\varphi'^{3/2}}\,d\varphi' = \frac{2}{\sqrt{\pi}}\,\frac{1}{\varphi^{*3/2}}\int_{\theta\left(\frac{\varphi^*}{\varphi^*-\varphi_0^*}\right)^{1/2}}^{\infty}\frac{\beta^3 e^{-\beta^2}\,d\beta}{(\beta^2-\theta^2)^{3/2}}$$

$$\cong \frac{2}{\sqrt{\pi}}\,\frac{1}{\varphi^{*3/2}}\left[e^{-\theta^2}\,\theta\,\frac{\varphi^{*1/2}}{\varphi_0^{*1/2}} + 2\int_{\theta\left(\frac{\varphi^*}{\varphi^*-\varphi_0^*}\right)^{1/2}}^{\infty}\frac{\beta e^{-\beta^2}(1-\beta^2)\,d\beta}{(\beta^2-\theta^2)^{1/2}}\right].$$

Inasmuch as the integral of the above-indicated second term is finite for $\beta = \theta$, then it is permissible to substitute for the lower limit the value θ, provided $\varphi^* \gg \varphi_0^*$. When this is done, it will be seen then that

$$\int_{\theta}^{\infty}\frac{\beta(1-\beta^2)e^{-\beta^2}}{(\beta^2-\theta^2)^{1/2}}\,d\beta = \frac{\sqrt{\pi}}{4}\,e^{-\theta^2}\,(1-2\theta^2)$$

and consequently, the sought evaluation for the third term in this series expansion under examination is given as

$$I_{-3} = A^*_{-3}\left[\frac{e^{-\theta^2}(1-2\theta^2)}{\varphi^{*3/2}} + \frac{2}{\sqrt{\pi}}\,\frac{1}{\varphi_0^{*1/2}}\,\frac{\theta e^{-\theta^2}}{\varphi^*}\right].$$

APPENDIX C

It is easy and illuminating to make a comparison between the values of β_e computed by assuming that the velocity profile for the flow in the boundary layer is that which corresponds to the outer-region solution (as contrasted to the correct expression for u) in the instances where the pressure gradient is of the type considered by Falker and Skan, which is $\frac{u_e}{u_\infty} = hx^m$. The comparison will be made here in only the simplest cases, for which $m = 0$ or $m = 1$.

Consider the case where $m = 0$ (thus where h must be taken as unity). For this particular situation, and under the assumption of incompressible flow, the pertinent basic relationships, in the present notation, are

$$\varphi^* = x \quad ; \quad \frac{Z_0}{u_\infty^2} = \frac{1}{2} \quad ; \quad \frac{Z}{u_\infty^2} = \frac{1}{2}\, \mathrm{erfc}\left(\frac{\bar{\psi}^*}{\sqrt{\varphi^*}}\right).$$

Let the usual arbitrary definition be made that the thickness of the boundary layer is that distance of displacement from the wall where the velocity ratio has reached 99.5 percent of the free-stream speed, or for which $\frac{u}{u_e} = 0.995$. Thus,

$$\frac{(Z)_{\bar{\varphi}=\bar{\varphi}_\delta}}{u_\infty^2} = \frac{0.01}{2}$$

so that $\mathrm{erfc}\,(\theta_\delta) = 0.01$, and thus one sees from the numerical tables that $\theta_\delta = 1.8$. Consequently, the angular inclination reached at the outer edge of the boundary layer is given in this instance as

$$\beta_e = \frac{2}{\sqrt{R_e}} \frac{\partial}{\partial \varphi^*}\left[\sqrt{\varphi^*}\int_0^\infty \left(\frac{1}{\sqrt{erf\,(\theta)}} - 1\right) d\,\theta\right] = \frac{1}{\sqrt{R_e}}\, 0.9673 \,.$$

In actuality the accepted value for the angular deviation is

$$\beta_e = \frac{0.865}{\sqrt{R_e}}$$

so that the error made in using the formula developed here is about 11.8 percent (above the value given by exact analysis).

Continuing with examination of the case where $m = 1$ (where h may have any value whatever), one may write, in the presently agreed-upon notation, that

$$\varphi^* = h\,\frac{x^2}{2}\ ;\ \frac{Z}{u_\infty^2} = h\,\varphi^*\ ;\ Z = Z_0\,\Gamma(2)\,u\,i^2\,erfc\,(\theta) = u_\infty^2\ h\,\varphi^*\,u\,i^2\,erfc\,(\theta)$$

Consequently, it may be seen in this case that $0.01 = hi^2 erfc(\theta_\delta)$ and the numerical tables thus show that $\theta_\delta = 1.46$.

Therefore, in this case, the angular inclination reached at the outer edge of the boundary layer is given as

$$\beta_e = \frac{2}{\sqrt{Re}}\,h\,x\ \ \frac{\partial}{\partial\,\varphi^*}\left[\sqrt{\varphi^*}\int_0^\infty \frac{1}{\sqrt{2\,\varphi^*}}\left(\frac{u_e}{u} - 1\right)d\,\theta\right] + h\,x\ \frac{\partial}{\partial\,\varphi^*}\left(\frac{u_\infty}{u_e}\right)\frac{2}{\sqrt{Re}}\sqrt{\varphi^*}\,\theta_\delta$$

$$= h\,x\ \frac{\partial}{\partial\,\varphi^*}\left(\frac{1}{\sqrt{2\,h\,\varphi^*}}\right)\frac{2\sqrt{h}\ x}{\sqrt{2}\,\sqrt{Re}}\,1.46 = -\ \frac{2.06}{\sqrt{Re}}\ \frac{1}{h^{1/2}\ x}$$

while the exact result given by the well-substantiated Hartree analysis is

$$\beta_e = -\ \frac{2}{\sqrt{Re}}\ \frac{1}{h^{1/2}\ x}\ .$$

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TABLE I

	$M_\infty = 5$	$M_\infty = 6$	$M_\infty = 8$
$Re = 10^6$	$x_e' = 0.435$ $x_e = .13986$	$x_e' = 0.522$ $x_e = .21092$	$x_e' = 0.696$ $x_e = .39425$
$Re = 10^7$	$x_e' = 0.435$ $x_e = .04423$	$x_e' = 0.522$ $x_e = .06670$	$x_e' = 0.696$ $x_e = .12467$

TABLE II

$K^*_{1,-3} = 4.43$	-------	-------	-------	-------	-------	-------	-------	------
$K^*_{1,-2} = .54$	-------	-------	$K_{2,0,-2} = -6.42$	-------	$K_{2,2,-2} = -4.53$	-------	$K_{2,4,-2} = -3.49$	------
------------	-------	$K_{2,-1,-1} = 8.74$	$K_{2,0,-1} = 8.82$	$K_{2,1,-1} = 8.87$	$K_{2,2,-1} = 8.90$	$K_{2,3,-1} = 8.92$	$K_{2,4,-1} = 8.95$	------
$K^*_{1,0} = 4.69$	$K_{2,-2,0} = 7.77$	$K_{2,-1,0} = 5.33$	$K_{2,0,0} = 5.86$	$K_{2,1,0} = 6.14$	$K_{2,2,0} = 6.33$	$K_{2,3,0} = 6.48$	$K_{2,4,0} = 6.61$	$K_{2,5,0} = 6.72$
$K^*_{1,1} = 4.24$	-------	$K_{2,-1,1} = 3.69$	$K_{2,0,1} = 4.45$	$K_{2,1,1} = 4.87$	$K_{2,2,1} = 5.12$	$K_{2,3,1} = 5.34$	$K_{2,4,1} = 5.50$	------
$K^*_{1,2} = 3.95$	$K_{2,-2,2} = 6.19$	$K_{2,-1,2} = 2.82$	$K_{2,0,2} = 3.53$	$K_{2,1,2} = 3.90$	$K_{2,2,2} = 4.15$	$K_{2,3,2} = 4.35$	$K_{2,4,2} = 4.50$	------
$K^*_{1,3} = 3.72$	-------	$K_{2,-1,3} = 1.79$	$K_{2,0,3} = 2.83$	$K_{2,1,3} = 3.61$	$K_{2,2,3} = 3.75$	$K_{2,3,3} = 4.05$	-------	------
$K^*_{1,4} = 3.55$	$K_{1,-2,4} = 6.35$	$K_{2,-1,4} = 1.12$	$K_{2,0,4} = 2.27$	$K_{2,1,4} = 3.09$	$K_{2,2,4} = 3.28$	-------	-------	------
$K^*_{1,5} = 3.35$	-------	-------	$K_{2,0,5} = -1.66$	-------	-------	-------	-------	------

$K_0 = 1.34$

TABLE III

M_∞	Re	$\frac{A_0^*}{A_0}$	$\frac{A_2^*}{A_0}$	$\frac{A_4^*}{A_0}$	$\frac{A_{-1}^*}{A_0}$	$\frac{A_1^*}{A_0}$	$\frac{A_{-2}^*}{A_0}$	$\frac{A_{-3}^*}{A_0}$	$\frac{A_3^*}{A_0}$	$\frac{A_5^*}{A_0}$
5	10^6	$-0.15\overline{3}$	$0.27\overline{5}$	$0.19\overline{5}$	$-0.009\overline{3}$	$-0.01\overline{5}$	$-0.0000\overline{6}$	$-0.000001\overline{1}$	$-0.01\overline{5}$	$0.01\overline{6}$
	10^7	$-.153\overline{9}$	$.27\overline{3}$	$.19\overline{4}$	$-.0029\overline{3}$	$-.005\overline{1}$	$-.000006$	$-.0000005$	$-.005\overline{8}$	$.02\overline{2}$
6	10^6	-0.179	$0.31\overline{5}$	$0.22\overline{5}$	-0.015	-0.0285	$-0.00010\overline{5}$	-0.000039	-0.033	0.019
	10^7	$-.180$	$.313$	$.23\overline{2}$	$-.0048$	$-.0089$	$-.00001$	$-.000003$	$-.007$	$.016$
8	10^6	$-0.22\overline{5}$	$0.40\overline{3}$	$0.29\overline{5}$	$-0.03\overline{4}$	-0.07	-0.0000008	----------	-0.11	0.005
	10^7	$-.23$	$.38\overline{5}$	$.29\overline{5}$	$-.010$	$-.021$	$-.00005$	$-.000007$	$-.02\overline{7}$	$.05\overline{5}$

TABLE IV

φ^*	$M_\infty = 5$; Re $= 10^6$		$M_\infty = 5$; Re $= 10^7$		$M_\infty = 6$; Re $= 10^6$		$M_\infty = 6$; Re $= 10^7$		$M_\infty = 8$; Re $= 10^6$		$M_\infty = 8$; Re $= 10^7$	
	x	y	x	y	x	y	x	y	x	y	x	y
0.1	0.0777	0.006035	0.09623	0.007455	0.05916	0.004603	0.09300	0.007199	0.02992	0.002426	0.07812	0.006047
.15	.12862	.009277	.1518	.010994	.10768	.007688	.14996	.010825	.06619	.004724	.13386	.009590
.20	.18382	.01224	.21140	.014195	.16145	.01057	.21197	.014153	.10952	.00704	.19656	.012950
.30	.30790	.016849	.34435	.019138	.28527	.15152	.35327	.019386	.21619	.01094	.34597	.018439
.35	.37731	.18318	.41842	.020706	.35591	.016645	.43349	.021082	.27994	.012287	.43417	.020297
.40	.45233	.19029	.49838	.021464	.43319	.017374	.52123	.021910	.35147	.012957	.53323	.021224
.50	.6175	.01746	.67891	.019698	.61098	.015609	.72334	.01901	.52180	.011222	.76466	.018920

TABLE V

φ^*	$M_\infty = 5.0$		$M_\infty = 6.0$		$M_\infty = 8.0$	
	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$
	p_e/p_∞		p_e/p_∞		p_e/p_∞	
0.05	2.04902	1.79776	2.64500	2.07217	5.2742	0.8977
.10	1.83217	1.66667	2.23414	1.8783	3.79635	2.4807
.15	1.68407	1.54400	2.0004	1.724	3.16168	2.1965
.20	1.55860	1.44906	1.81752	1.575	2.74123	1.959
.30	1.33905	1.25376	1.51745	1.335	2.14587	1.562
.35	1.23993	1.16282	1.38773	1.222	1.91334	1.392
.40	1.14653	1.07631	1.26791	1.116	1.70923	1.238
.50	1.02448	.91667	1.0553	.927	1.366	1.026

TABLE VI

φ^*	$M_\infty = 5.0$		$M_\infty = 6.0$		$M_\infty = 8.0$	
	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$
	θ_δ		θ_δ		θ_δ	
0.05	1.7946	1.8110	1.7807	1.8029	1.5908	1.7871
.10	1.7970	1.8101	1.7830	1.8033	1.6602	1.7896
.15	1.7967	1.8076	1.7840	1.8007	1.6875	1.7888
.20	1.7954	1.8038	1.7835	1.7980	1.7015	1.7861
.30	1.7908	1.7965	1.7800	1.7911	1.7198	1.7782
.35	1.7876	1.7933	1.7769	1.7870	1.7230	1.7732
.40	1.7845	1.7896	1.7740	1.7830	1.7230	1.7680
.50	1.7775	1.7830	1.7663	1.7740	1.72	1.7560

TABLE VII

φ^*	$M_\infty = 5.0$		$M_\infty = 6.0$		$M_\infty = 8.0$	
	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$
	φ		φ		φ	
0.008	0.047775	0.47996	0.047593	0.047889	0.044986	0.047678
.05	.10525	.104765	.105652	.104999	.111775	.105462
.10	.176984	.176203	.177663	.176583	.187172	.177341
.15	.239846	.238855	.240758	.239354	.252303	.240335
.20	.297592	.296444	.298701	.297046	.311755	.298217
.30	.403428	.402071	.404870	.402838	.420111	.404374
.35	.45295	.451515	.454541	.452362	.470599	.454084
.40	.500771	.499264	.502505	.500197	.519289	.502113
.50	.592316	.590672	.594332	.591807	.612406	.594151

TABLE VIII

φ^*	$M_\infty = 5.0$		$M_\infty = 6.0$		$M_\infty = 8.0$	
	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$	$Re = 10^6$	$Re = 10^7$
	$\sqrt{Re}\ \overline{\varphi}^{*1/2} C_f$		$\sqrt{Re}\ \overline{\varphi}^{*1/2} C_f$		$\sqrt{Re}\ \overline{\varphi}^{*1/2} C_f$	
0.10	0.4931	0.455	0.507	0.426	0.466	0.385
.15	.4935	.458	.505	.428	.464	.386
.20	.4973	.463	.508	.433	.467	.391
.30	.5152	.483	.524	.454	.48	.414
.35	.53	.499	.539	.472	.494	.433
.40	.551	.521	.560	.495	.514	.458
.50	.61	.583	.621	.561	.583	.535

TABLE IX

M_∞	Re	x_e	$\frac{A^*_{-1}}{A_0}$	$\frac{A^*_{-2}}{A_0}$	$\frac{A^*_{-3}}{A_0}$
5	10^6	0.13986	-0.007496	-0.0001546	-0.0000655
	10^7	.04423	-.002370	-.0000154	----------
6	10^6	0.21093	-0.011305	-0.0003515	-0.000152
	10^7	.0667	-.003575	-.0000351	-.0000143
8	10^6	0.39425	-0.0211318	-0.001228	-0.000557
	10^7	.12467	-.006682	-.0001222	-.0000518

TABLE X

φ^*	M_∞ = 5.0		M_∞ = 6.0		M_∞ = 8.0	
	Re = 10^6	Re = 10^7	Re = 10^6	Re = 10^7	Re = 10^6	Re = 10^7
	x		x		x	
0.10	0.13547	0.16949	0.12745	0.18453	0.1230	0.20486
.15	.21743	.25758	.21652	.28306	.22110	.32523
.20	.30074	.34677	.30809	.38227	.34513	.44729
.30	.46942	.52521	.49428	.58173	.55995	.69406
.35	.55449	.61446	.58870	.68184	.67091	.81837
.40	.63988	.70371	.68403	.78210	.78335	.94309
.50	.81135	.88223	.87611	.98299	-------	-------

TABLE XI

φ^*	$M_\infty = 5.0$		$M_\infty = 6.0$		$M_\infty = 8.0$	
	Re = 10^6	Re = 10^7	Re = 10^6	Re = 10^7	Re = 10^6	Re = 10^7
	θ_δ		θ_δ		θ_δ	
0.05	1.8252	1.8195	1.84449	1.8190	1.9625	1.8227
.10	1.8156	1.8208	1.8144	1.8186	1.8321	1.8152
.15	1.8140	1.8215	1.8101	1.8195	1.8065	1.8150
.20	1.8145	1.8219	1.8098	1.8199	1.7997	1.8156
.30	1.8157	1.8224	1.8109	1.8207	1.7984	1.8168
.35	1.8163	1.8226	1.8119	1.8210	1.7988	1.8173
.40	1.8168	1.8228	1.8122	1.8212	1.7994	1.8182
.50	1.8175	1.8230	1.8133	1.8216	1.8009	1.7184

TABLE XII

φ^*	$M_\infty = 5.0$				$M_\infty = 6.0$				$M_\infty = 8.0$			
	Re = 10^6		Re = 10^7		Re = 10^6		Re = 10^7		Re = 10^6		Re = 10^7	
	p_e/p_∞	A	p_e/p_∞	A	p_e/p_∞	A	p_e/p_∞	A	p_e/p_∞	A	p_e/p_∞	A
0.10	1.1020	0.5002	1.0272	0.5146	1.1709	0.5517	1.0436	0.4759	1.3757	0.5358	1.0887	0.4164
.15	1.0790	.5148	1.0022	.5174	1.1294	.5594	1.0348	.4804	1.2752	.5743	1.0693	.4266
.20	1.0667	.5229	1.0019	.5194	1.1074	.5612	1.0297	.4828	1.2401	.6141	1.0586	.432
.30	1.0528	.5312	1.0015	.5213	1.0926	.5588	1.0239	.4855	1.1795	.6292	1.0466	.4337
.35	1.0450	.5345	1.0014	.5221	1.0765	.5572	1.0221	.4864	1.1617	.6356	1.0428	.4335
.40	1.045	.5361	1.0014	.5224	1.0708	.5560	1.0206	.4872	1.1481	.6399	1.0397	.4368
.50	1.04	.5389	1.0012	.5228	1.0623	.5534	1.0183	.4881	1.1284	.6472	1.0353	.439

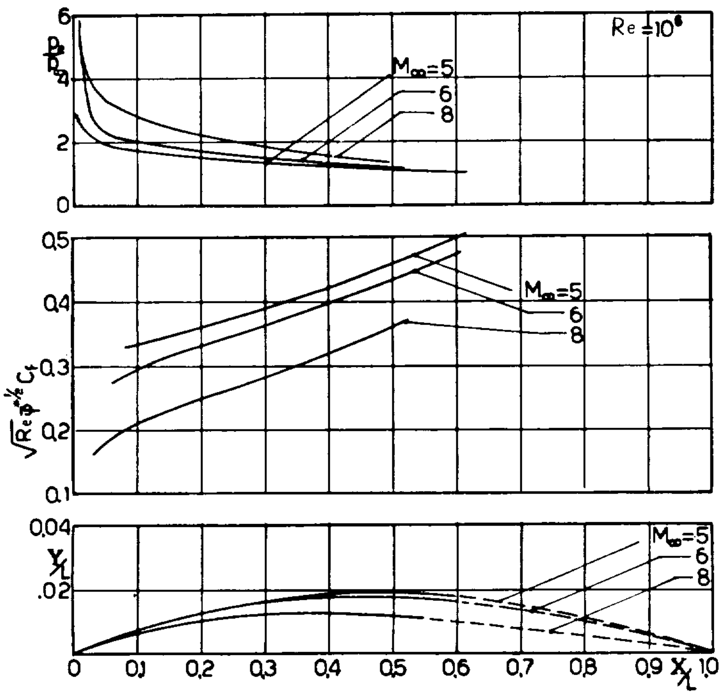

Figure 1.- Curved airfoil profiles. Re = 10^6.

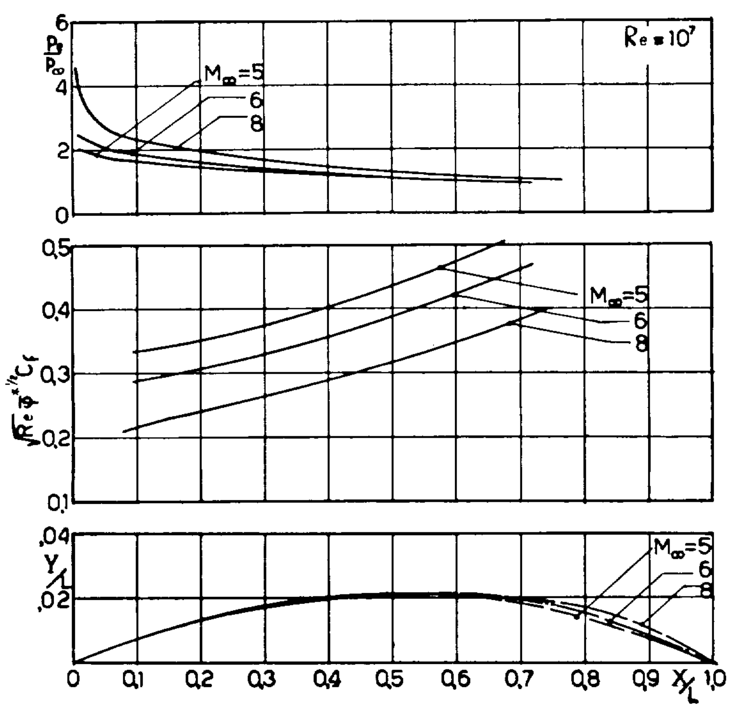

Figure 2.- Curved airfoil profiles. Re = 10^7.

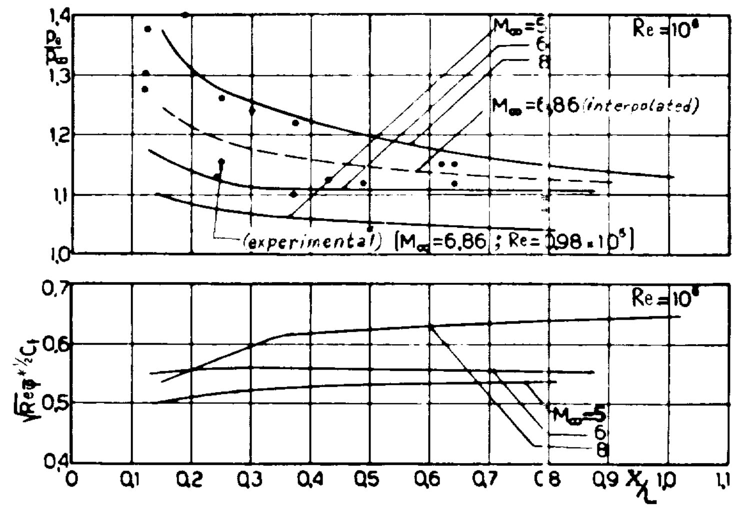

Figure 3.- Flat-plate airfoil. Re = 10^6.

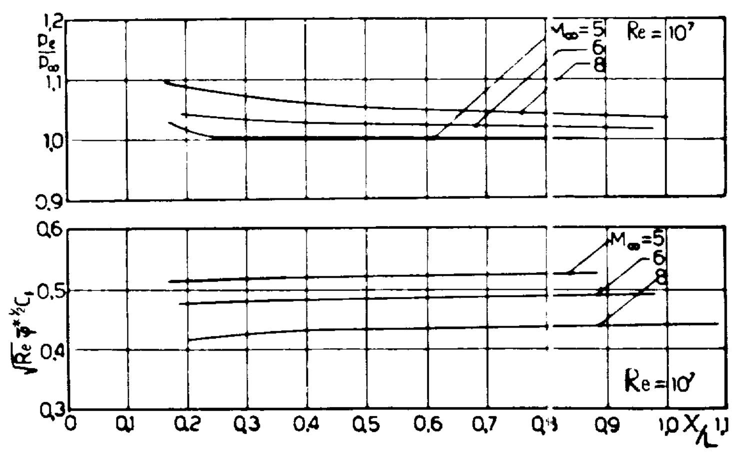

Figure 4.- Flat-plate airfoil. Re = 10^7.